A STUDY OF TORSION THEORIES

BY

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ABSTRACT

This paper is a study of torsion theories and related topics such as pure and divisible for both integral domains and non-integral domains. Motivating the study was a super goal of investigating the existence and uniqueness of torsion-free covering modules over not necessarily integral domains. These are shown to exist and uniquely in the first section of the paper for the integral domain case. Two torsion theories for not necessarily integral domains are studied in the second and third sections. In the second section Lawrence Levy's theory is studied and it is proved that the set of torsion elements of a module forms a submodule, if and only if the ring has a right quotient ring. In the third section Akira Hattori's theory is studied and it is shown that the two theories agree where both are defined and that in the case of integral domains they both agree with the usual torsion theory. In the third section homology is used considerably in both definitions and proofs.

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Torsion theory as well as the related properties of pure and divisible are studied in three settings. In the first situation, commutative integral domains, Enoch's paper (2), Torsion-Free Covering Modules, is studied. In this the usual definitions of torsion and divisible are used, and "torsion-free covering modules" are shown to exist and uniquely. Here, the property of torsion was not so much studied as was an interesting result of the property in the integral domain situation.

In the second situation, rings in general, the usual definitions as in the integral domain case yield each module of a ring torsion unless the ring has no zero divisors. This leads to a reformulation of definitions and to a study of Levy's paper (5), Torsion-Free and Divisible Modules Over Non-Integral Domains. Taking Levy's definitions of torsion and divisible, it is shown that the torsion elements of a module form a submodule if and only if the ring has a quotient ring.

In the third situation, rings with unit, Hattori's paper (3), A Foundation of Torsion Theory for Modules Over General Rings is studied. In this paper still other definitions of torsion and divisible are made and homology is used extensively in developing some of the results of the torsion property.

At the end of the sections on Levy's paper and Hattori's paper comparisons of the theories are made. This is to say that Levy's and Hattori's definitions are equivalent under certain conditions and that both theories are equivalent to the usual theory in the integral domain case.

For needed definitions and results in homology, Cartan and Eilenberg's (1), Homological Algebra and Jans' (4) Rings and Homology were used as references. For properties concerning injectives and divisible Cartan and Eilenberg (1) was used.

Section 1.

Through out this section, unless otherwise stated, A will be considered to be a commutative integral domain with identity and K its field of fractions.

Definition: An A-module E is said to be torsion-free if $\alpha x = 0$ for $\alpha \varepsilon A$, $x \varepsilon E$ implies $\alpha = 0$ or $x = 0$.

Definition: A submodule E_1 of an A-module E is pure in E if $\alpha E_1 = \alpha E \cap E_1$ for all $\alpha \varepsilon A$.

<u>Proposition 1.1</u>: If E is torsion-free, a submodule E_1 of E is pure in E iff E/E_1 is torsion-free.

Proof: Suppose E_1 is pure in E, Then $\alpha E_1 = \alpha E \cap E_1$ for all $\alpha \varepsilon A$. Now if $\alpha(e + E_1) = 0$, then $\alpha e + E_1 = 0$ so $\alpha e \varepsilon E_1$. Hence, since E_1 pure in E and $\alpha e \; \varepsilon \; \alpha E \cap E_1$, $\alpha e \; \varepsilon \; \alpha E_1$ so $\alpha e = \alpha e_1$ for some $e_1 \; \varepsilon \; E_1$. Thus, $\alpha(e - e_1) = 0$. But $e - e_1 \; \varepsilon \; E$ and since E torsion-free $\alpha = 0$ or $e - e_1 = 0$, that is $\alpha = 0$ or $e = e_1$ which implies $\alpha = 0$ or $e \varepsilon E_1$. Thus $\alpha = 0$ or $e + E_1 = 0$. Now suppose if E/E_1 is torsion-free, then for each $\alpha \; \varepsilon \; A$, $e \varepsilon E$, $\alpha(e + E_1) = 0$ implies $\alpha = 0$ or $e \varepsilon E_1$. Clearly $\alpha E_1 \subset \alpha E \cap E_1$ as E_1 is an A submodule of E. Now let $e \; \varepsilon \; \alpha E \cap E_1$, then $e = \alpha e'$ for some $e' \; \varepsilon \; E$ and so $\alpha e' + E_1 = 0$ as $\alpha e' \; \varepsilon \; E_1$. This implies $\alpha(e' + E_1)$ is equal to 0 and hence, $\alpha = 0$ or $e' \; \varepsilon \; E_1$. If $\alpha = 0$ $\alpha e' \; \varepsilon \; \alpha E_1$ as

$\alpha e' = 0 \in E_1$. If $e' \in E_1$, $\alpha e' \in \alpha E_1$ and so in either case $\alpha e' \in \alpha E_1$, so $\alpha e' = e \in \alpha E_1$. Thus $\alpha E \cap E_1 \subset \alpha E_1$ for each α in A. So $\alpha E_1 = \alpha E \cap E_1$ and E_1 is pure in E.

Proposition 1.2: The union of a chain of pure submodules of an A-module E is a pure submodule of E.

Proof: Let $(E_\lambda)_{\lambda \varepsilon L}$ be a family of pure submodules which form a chain indexed by an appropriate set L. Then the union of these submodules is again a submodule as they are totally ordered by inclusion. Now if $e \in \alpha E \cap \bigcup_{\lambda \varepsilon L} E_\lambda$ then $e \in E_\lambda$ for some $\lambda \in L$, so $e \in \alpha E \cap E_\lambda$ and since E_λ is a pure submodule of E $e \in \alpha E_\lambda$. But $\alpha E_\lambda \subset \alpha \bigcup_{\lambda \varepsilon L} E$ so $e \in \alpha \bigcup_{\lambda \varepsilon L} E_\lambda$ so $\alpha E \cap \bigcup E_\lambda = \alpha \bigcup E_\lambda$ and so the union is pure in E.

Proposition 1.3: If $E_2 \subset E_1$ are submodules of E such that E_2 is pure in E and E_1/E_2 is pure in E/E_2 then E_1 is pure in E.

Proof: If E_2 is pure in E then $\alpha E_2 = \alpha E \cap E_2$ for all $\alpha \in A$. And also if E_1/E_2 is pure in E/E_2 then $\alpha E_1/E_2 = \alpha E/E_2 \cap E_1/E_2$ for all $\alpha \in A$. Clearly $\alpha E_1 \subset \alpha E \cap E_1$ so let $\alpha e \in E_1$ for some $e \in E$ and $\alpha \in A$. Then $\alpha e + E_2 = \alpha e_1 + E_2$ for some $e_1 \in E_1$, and so $\alpha e - \alpha e_1 \in E_2$ which can be written $\alpha(e - e_1) \in E_2$ but since E_2 pure in E, $\alpha(e - e_1) = \alpha e_2$ for some e_2 in E_2. But then $\alpha e = \alpha(e_1 + e_2)$ and since $e_1 + e_2 \in E_1$, $\alpha(e_1 + e_2) \in \alpha E_1$ so $\alpha e \in \alpha E_1$ and we have $\alpha E \cap E_1 \subset \alpha E_1$. So for all $\alpha \in A$, $\alpha E_1 = \alpha E \cap E_1$ and E_1 is pure in E.

<u>Proposition 1.4</u>: For any A-module E there exists a torsion-free A-module E_1, and a surjection p: $E \to E_1$, such that if ϕ is any A-linear mapping from E into a torsion-free module F then there is a unique linear mapping f: $E_1 \to F$ such that $f \circ p = \phi$. i.e., the diagram:

$$\begin{array}{ccc} E & \xrightarrow{p} & E_1 \\ & \searrow^{\phi} & \downarrow f \\ & & F \end{array}$$

commutes.

Proof: Let E_1 be E/E' where E' is the torsion sub-module of E. (The set of all elements in E that are annihilated by a non-zero element of A is a submodule of E and E/E' is torsion free.) Let p be the connonical surjection p: $E \to E/E'$. Let ϕ be an A-linear mapping from E into a torsion-free module F. Then define f: $E_1 \to F$ by $f(\overline{x}) = \phi(x)$ if $x, x' \in \overline{x}$. There exists $\alpha \in A$ $\alpha \neq 0$ such that $\alpha(x - x') = 0$. So $\phi(\alpha(x - x')) = 0 = \alpha(\phi(x - x')) = \alpha[\phi(x) - \phi(x')]$ but since F torsion-free and $\alpha \neq 0$ $\phi(x) - \phi(x') = 0$ and hence, $\phi(x) = \phi(x')$ and the map is well defined. Show f is homomorphism. f is a homomorphism for $f(\overline{x} + \overline{y}) = \phi(x + y) = \phi(x) + \phi(y) = f(\overline{x}) + f(\overline{y})$, also $\alpha \neq 0$ $\alpha f(\overline{x}) = \alpha\phi(x) = \phi(\alpha x) = f(\alpha \overline{x}) = f(\alpha \overline{x})$. Clearly, $f \circ p = \phi$. Now if g is any map from $E_1 \to F$ such that $g \circ p = \phi$, then $g \circ p(x) = g(\overline{x}) = \phi(x) = f(\overline{x})$ and so $g = f$.

Now we will define torsion-free covering module and proceed to develop it. Essentially, we wish to reverse the diagram for the preceeding proposition and the positions of of E and E_1.

Definition: Given a module E then a torsion-free module T(E) and a map Ψ: T(E) $\to$ E will be called respectively a torsion-free covering module of E and a torsion-free covering of E if they satisfy the following:

(1) Ker Ψ contains no non-trivial pure submodules of E.

(2) if Ψ: F $\to$ E is a linear map where F is torsion-free then there exists a linear map f: F $\to$ T(E) such that $\Psi \circ f = \phi$.

This is to say that the diagram

$$\begin{array}{ccc} T(E) & \xrightarrow{\Psi} & E \\ & \nwarrow^{f} & \uparrow \phi \\ & & F \end{array}$$

commutes.

The existence and uniqueness of both of these are proved shortly but first we need to make several definitions and establish a few lemmas.

Definition: A linear map Ψ: E' $\to$ E will be said to have the torsion-free factor property abbreviated (TFF), if for any linear mapping ϕ: F $\to$ E, where F is torsion-free, there exists a linear map f: F $\to$ E' such that $\Psi \circ f = \phi$. i.e., the diagram:

$$\begin{array}{ccc} E' & \xrightarrow{\Psi} & E \\ & \nwarrow_{f} & \uparrow \phi \\ & & F \end{array}$$

commutes.

<u>Lemma 1.5</u>: If : E' $\to$ E has the torsion-free factor property and E_1 is a submodule of E then the linear mapping

$f\colon \Psi^{-1}(E_1) \to E_1$ which agrees with Ψ on $\Psi^{-1}(E_1)$ has the torsion-free factor property.

Proof: Define $f\colon \Psi^{-1}(E_1) \to E_1$ by $f(x) = \Psi(x)$. Let F be a torsion-free module and ϕ a linear mapping, $\phi\colon F \to E_1$. Now since Ψ had the torsion-free factor property there exists a linear mapping $g\colon F \to E'$ such that $\Psi \circ g = \phi$. Now $g/g^{-1}(\Psi^{-1}(E_1))$ is a linear map $g\colon F \to \Psi^{-1}(E_1)$ such that $f \circ g/g^{-1}(\Psi^{-1}(E_1)) = \phi$ so f has the torsion-free factor property.

Definition: An A-module M is said to be divisible if for each $m \in M$, $a \in A \setminus \{0\}$ there exists $m' \in M$ such that $m = am'$.

Remark 1.6: For integral domains injective modules are divisible modules.

Proof: Let M be an injective A-module where A is an integral domain. i.e., given any module E and a submodule E' and any homomorphism $\phi\colon E' \to M$ there exists a homomorphism $f\colon E \to M$ such that $f \circ i = \phi$ where i is the connonical injection. i.e., the diagram:

$$\begin{array}{ccccc} 0 & \to & E' & \to & E \\ & \phi & \downarrow & \swarrow & f \\ & & M & & \end{array}$$

commutes.

<u>Lemma 1.6'</u>: In order that a module M be injective it is necessary and sufficient that for each left ideal Λ of A and each homomorphism $f\colon \Lambda \to M$ there exists an element $g \in M$ such that $f(\lambda) = \lambda g$ for all $\lambda \in \Lambda$.

Proof: Suppose M injective, then the homomorphism f has an extension g: $A \to M$ and $f\lambda = g\lambda = g(1)$ for each $\lambda \in \Lambda$, and the conclusion is necessary. To prove sufficiency consider a module E, a submodule E', and a homomorphism f: $E' \to E$. Consider the family F of all pairs (E_1, f_1) where E_1 is a submodule of E containing E' and f_1: $E_1 \to E$ is an extension of f. We introduce a partial order in F by letting $(E_1, f_1) < (E_2, f_2)$ if $E_1 \subset E_2$, and f_2 is an extension of f_1. The family F is obviously inductive and therefore by Zorn's lemma there is an element (E_o, f_o) of F which is maximal. Now $E_o = E$ since, if not, suppose $x \in E$ and $x \notin E_o$. The set of all $\lambda \in A$ such that $\lambda x \in E_o$ forms a left ideal Λ of A and the map f_o': $\Lambda \to E$ defined by $f_o'(\lambda) = f_o(\lambda x)$ is a homomorphism.* There is therefore an element $g \in M$ such that $f_o(\lambda x) = \lambda g$ for all $\lambda \in \Lambda$. Setting $f_o'(e + \lambda x) = f_o e + \lambda g$, $e \in E_o$, $\lambda \in A$, yields then a map f_o'' of the submodule $E_o +$ x, of E which is an extension of f_o'. Thus (E_o, f_o) is not maximal.

Continuing with the proof of the remark. Let M be an injective module and let $m \in M$, $\lambda \in A$, $\lambda \neq 0$. Consider the ideal $\Lambda = \lambda A$. Since $\alpha\lambda = \beta\lambda$ implies $\alpha = \beta$ (integral domain) the formula $f(\alpha\lambda) = \alpha m$ defines a homomorphism f: $\Lambda \to M$. Since M is injective there exist by the preceeding lemma a $m' \in M$ with $f(\lambda) = \lambda m'$ for all $\lambda \in \Lambda$. Thus m =

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*Show f_o' is well-defined for if $\lambda = \beta$ then $f_o'(\lambda) = f_o(\lambda x)$ and $f_o'(\beta) = f_o(\beta x)$ but $f_o(\lambda x) = f_o(\beta x) = f_o(\lambda x - \beta x) = f_o((\lambda - \beta)x) = f_o(0x) = f_o(0) = 0$ so $f_o'(\lambda) = f_o'(\beta)$.

$f(\lambda) = \lambda m'$ so M is divisible.

Lemma 1.6: If E is injective then Ψ: $E' \to E$ has the torsion-free factor property if and only if for every linear map ϕ: $F \to E$, where F is torsion-free and injective there is a linear mapping f: $F \to E'$ such that $\Psi \circ f = \phi$.

Proof: By definition if Ψ: $E' \to E$ has the torsion-free factor property (TFF) then for every linear map ϕ: $F \to E$ where F is torsion-free there exists a linear mapping f: $F \to E'$ such that $\Psi \circ f = \phi$. If F is torsion-free and injective it is still torsion-free and the existence is still guaranteed if Ψ has TFF. Now if ϕ: $F_1 \to E$ is any linear mapping where F_1 is torsion-free, then since F_1 is a submodule of a torsion-free injective (hence divisible) module f $(F_1 \otimes K)$, and since E is injective, there exists a linear mapping ϕ: $F \to E$ such that $\phi/F_1 = \phi_1$. Then by hypothesis there exists f: $F \to E$ such that $\Psi \circ f = \phi$ and $\Psi \circ (f/E_1) = \phi_1$.

Lemma 1.7': (Theorem 3.3 pg. 9) Each module E is a submodule of an injective module.

Proof: For each module E we shall define a module D(E) containing E with the following property: (*) If Λ is a left ideal of A and f: $\Lambda \to E$, then there is an element $g \in D(E)$ such that $f(\lambda) = \lambda g$ for all $\lambda \in \Lambda$. Let Φ be the set of all pairs (Λ, f) formed by a left ideal Λ of A and a homomorphism f: $\Lambda \to E$. Let F be the free module generated by the elements of Φ. Let D(E) be the quotient of the direct sum $E + F_\Phi$ by the submodule generated by the elements

$$(f(\lambda), -\lambda(\Lambda, f)) \quad (\Lambda, f) \in \Phi, \lambda \in \Lambda$$

The mapping $e \to (e, o)$ yields a homomorphism Φ: $E \to D(E)$. If $\phi(e) = 0$ then $\phi(e) = (e, o) = \Sigma u_i(f_i(\lambda_i), -\lambda_i(\Lambda_i, f_i)) = \Sigma(f_i(u_i, \lambda_i), -u_i\lambda_i(\Lambda_i f_i))$. Therefore, $\Sigma u_i\lambda_i(\Lambda_i, f_i) = 0$ in f_Φ, which implies $e = 0$. Thus Φ is a monomorphism and, by identifying e and $\phi(e)$ we may regard E as a submodule of D(E).

We now prove that D(E) has the property (*). Let f: $\Lambda \to E$ where Λ is a left ideal in A. Then $(\Lambda, f) \in \Phi$. Let g be the image in D(E) of the element $(0, (\Lambda, f))$ of $E + F_\Phi$. Then for each $\lambda \in \Lambda$, $f(\lambda) = (f(\lambda), 0) = (0, \lambda(\lambda, f)) = \lambda g$ as required.

Now let Ω be the least infinite ordinal number whose cardinal is larger than that of the ring A. We define $Q\alpha(E)$ for $\alpha \leq \Omega$ by transfinite induction as follows: $Q_1(E) = D(E)$; if $\alpha = \beta + 1$ then $Q_\alpha(E) = D(Q_\beta(E))$; if α is a limiting ordinal then $Q_\alpha(E)$ is the union of Q_β with $\beta < \alpha$. We now prove that $Q_\Omega(E)$ is injective. Let f: $\Lambda \to Q_\Omega(E)$ where Λ is a left ideal of A. Then because of the choice of Ω we have $f(\Lambda) \subset Q_\alpha(E)$ for some $\alpha < \Omega$. Then by (*) there is an element $g \in D(Q_\alpha(E)) = Q_{\alpha+1}(E) \subset Q_\Omega(E)$ with $f(\lambda) = \lambda g$ for all $\lambda \in \Lambda$. Thus by lemma 1.6' $Q_\Omega(E)$ is injective.

<u>Lemma 1.7</u>: For every module E there exists a torsion-free module E' and a linear mapping Ψ: $E' \to E$ having TFF.

Proof: Using Lemma 1.5 and the preceeding lemma it suffices to assume that E is injective since if it were not we simply find an injective module that E is imbedded in, get the map with TFF, and then restrict the map to E. Then using Lemma 1.6 we see that in order to prove that a linear mapping Ψ: $E' \to E$ has TFF, it suffices to show that if ϕ: $F \to E$, where F is torsion free and injective, then there is a linear mapping f: $F \to E'$ such that $\Psi \circ f = \phi$.

Now every torsion-free injective A-module is a K module by the map defined by: if $\lambda \varepsilon A$, $a\lambda = a\lambda$ for all $a \varepsilon F$, F a torsion-free, injective A-module and if $\lambda = \lambda_1/\lambda_2$ where $\lambda_2 \neq 0$ $a\lambda_1/\lambda_2 = b\lambda_1$ where b is such that $a = b\lambda_2$. b is uniquely determined as F torsion-free and injective. Also every K-module is the direct sum of a family of submodules isomorphic to K, since, if M is a K-module, then M is a K vector space and hence isomorphic to $\bigoplus_{\lambda \varepsilon L} K\lambda$ where L is the dimension of M. From what we have said, it suffices to show that there exists E' and Ψ: $E' \to E$, Ψ A-linear, and for each A-linear map ϕ': $M \to E$ there exists an A-linear map f': $M \to E'$ such that $\Psi \circ f' = \phi'$. But since $M = \bigoplus_{\lambda \varepsilon L} K$ and since for each map ϕ: $\bigoplus_{\lambda \varepsilon L} K_\lambda \to E$ $\phi(K\lambda)_{\lambda \varepsilon L} = \sum_{\lambda \varepsilon L} \phi_\lambda(K\lambda)$ it suffices to show that for each ϕ: $K \to E$ there exists an A-linear map f': $K \to E'$ such that $\Psi \circ f' = \phi'$ i.e., $F = K$ in *, $= \bigoplus K_\phi$ in *, $\phi \varepsilon$ Hom (K, E). Now let $E' = \bigoplus K_\phi$ and $\phi \varepsilon$ Hom (K, E) define Ψ: $\bigoplus K_\phi \to E$, $\phi \varepsilon$ Hom (K, E) by $\Psi(k_\phi)$ $\phi \varepsilon$ Hom (K, E) $= \Sigma\, \phi(K_\phi)$, $\phi \varepsilon$ Hom (K, E). Then for each A-linear map ϕ': $K \to E$ there exists an A-linear map f': $K \to E'$. Namely define f' (k) = (k_ϕ) $\phi \varepsilon$ Hom (K, E) where $k_\phi = 0$, $\phi \neq \phi'$ and $k_\phi = k$. Clearly

$\Psi \circ f' = \phi'$ and the proof is completed.

Lemma 1.8: If ψ: $E' \to E$ has TFF and N is a submodule of E' contained in the Kernel of Ψ, then the induced mapping Ψ', $E'/N \to E$ has TFF.

Clearly Ψ' exists and is homomorphism. Now let ϕ: $F \to E$ be any A-linear mapping where F is torsion-free. Then there exists an A-linear map f: $F \to E'$ such that $\Psi \circ f = \phi$, Let f': $F \to E'/N$ be the composition of f and the cononical surjection. f' is a homomorphism. Now then $\Psi' \circ f' = \phi$ since $\Psi' \circ f'(X)) = \Psi'(f(x)+N) = \Psi(f(x))$. But since $\Psi \circ f = \phi$, $\Psi(f(X)) = \phi(x)$.

Remark 1.9: If Ψ: $E' \to E$ has TFF where E' is torsion free and N is a maximal element among the pure submodules of E' contained in the kernel of Ψ, then the induced mapping Ψ' of E'/N is a torsion-free covering of E. i.e.

(1) Ker Ψ' contains no nontrivial, pure submodule of E.

(2) if ϕ: $F \to E$ is a linear mapping with F torsion-free, there exists a linear map f: $F \to E'/N$ such that $\Psi' \circ f' = \phi$.

Proof: By Lemma 1.8 Ψ' has TFF. Ker Ψ' contains no nontrivial, pure submodule of E' since if E_1 is a nontrivial pure submodule of E contained in Ker Ψ', then there exists a submodule $E'' \neq N$ of E' such that $E''/N \simeq E_1$ and E"/N is pure in E'. N pure in E' implies E" pure in E' with

$N \subset E''$ a contradiction to N being maximal among the pure submodules of E' contained in the Ker Ψ.

<u>Theorem 1.10</u>: Every module E has a torsion-free covering.

Proof: Every module E has a torsion-free module E' and a linear mapping Ψ: $E' \to E$ having TFF by Lemma 1.7. Let N be maximal among pure submodules of E' contained in Ker Ψ. Then the induced mapping Ψ': $E'/N \to E$ has TFF by Lemma 1.8 and by Remark 1.9 Ψ' is a torsion-free covering of E and E'/N is a torsion-free covering module of E.

Having established the existence we would want torsion free coverings to be unique.

<u>Theorem 1.11</u>: If Ψ': $E' \to E$ and Ψ'': $E'' \to E$ are two torsion-free coverings of E and f: $E' \to E''$ is a linear mapping such that $\Psi'' \circ f = \Psi'$, then f is an isomorphism.

Proof: Since Ψ'' is a torsion-free covering of E, there exists a linear mapping f: $E' \to E''$ such that $\Psi'' \circ f = \Psi'$. But then Ker f is a pure submodule of E' (since E" is torsion-free) which is contained in ker Ψ! But since Ψ' is a torsion-free covering, ker f is 0. Thus f is a monomorphism and so card (E') $\leq$ card (E"). Similarly card (E") $\leq$ card (E') so card (E") = card (E'), that is, all torsion-free covering modules of E have the same cardinality. Thus let X be a set containing the elements of E' and E" and such that card (X) > card (E'). Let F be the set of pairs (E_o, Ψ_o), where E_o is an A-module whose elements are elements of X, and where Ψ_o is a linear mapping $E_o \to E$, which is a torsion-free covering of E. Then (E', Ψ') and (E", Ψ") belong to F.

Partially order F by setting $(E_o, \Psi_o) \leq (E_1, \Psi_1)$, if E_o is a submodule of E_1 and $\Psi_1/E_o = \Psi_o$. Then F has maximal elements for if C is a chain of F let E* be the union of the first cordinates of the pairs in C with the unique structure of an A-module such that E_o is a submodule of E* for each (E_o, Ψ_o) in C and let Ψ^*: $E^* \to E$ be the unique linear mapping such that $\Psi^*/E_o = \Psi_o$ for each pair (E_o, Ψ_o) in C.

Then Ψ^* clearly has the torsion-free factor property. If N is a pure submodule of E* contained in Kernel Ψ^* then N E_o is a pure submodule of E_o contained in Kernel Ψ_o for each (E_o, Ψ_o) in C. Thus N $E_o = 0$ for each (E_o, Ψ_o) in C so N = 0. Thus, (E*, Ψ*) belongs to F. Clearly (E*, Ψ*) is an upper bound of C.

Thus assume (E*, Ψ*) is a maximal element of F. Now let f_1: $E^* \to E'$ be any linear mapping such that $\Psi' \circ f_1 = \Psi^*$. By previous remarks we know f_1 is a monomorphism. We would like to show that it is also an epimorphism. Let $Y \subset X$ be such that card (Y) = card (E' - f_1 (E*)) and such that $E^* \cap Y = \phi$.

Such a Y is available since card (X) > card (E') = card (E*). Let $E_o = E^* \cup Y$ and let g be a bijection $E_o \to E'$ such that $g/E^* - f_1$ and g (Y) = E' - f_1 (E*). Then E_o can be made uniquely into an A-module so that g becomes an isomorphism. Letting E_o denote this module we see that E* is a submodule of E_o, that $(E_o, \Psi' \circ g)$ is an element of F and $\Psi' \circ g/E^* = \Psi' \circ f_1 = \Psi^*$ so that $(E^*, \Psi^*) \leq (E_o, \Psi' \circ g)$. But (E^*, Ψ^*) is a maximal element of F, hence, $= \phi$, so $E' - f_1 (E^*) = \phi$ or f_1 is an epimorphism. Similarly any linear mapping f_2: $E^* \to E''$ such that $\Psi'' \circ f_2 = \Psi^*$ is an epimorphism. But $f \circ f_1$ is such a mapping since $\Psi'' \circ f \circ f_1 = \Psi' \circ f_1 = \Psi^*$, hence, $f \circ f_1$ is an epimorphism but then f must be an epimorphism. But f is a monomorphism, hence an isomorphism.

Theorem 1.12: If Ψ: $T(E) \to E$ is a torsion-free covering of E with Kernel G then the sequence $0 \to \mathrm{Ext}_A^N (F, G) \to \mathrm{Ext}_A^N (F, T(E)) \to \mathrm{Ext}_A^N (F, E) \to 0$ is exact if F is torsion-free and if $n \geq 1$.

Proof: By definition of T(E), $\mathrm{Hom}(F, T(E)) \to \mathrm{Hom}(F, E) \to 0$ is exact whenever F is torsion-free. Choose $0 \to K \to L \to F \to 0$ exact with L a free module. Then $\mathrm{Ext}^i (K, \_\_) \cong \mathrm{Ext}^{i+1} (F, \_\_)$ naturally. (Every free module is projective (1, pg. 7), and if $0 \to A \to P \to B \to 0$ is exact with P projective then $\mathrm{Ext}^n (A, C) = \mathrm{Ext}^{n+1} (B, C)$ for all C and all $n \geq 1$ (4 pg. 47).

By the exact sequence in the 1st variable of Ext Theorem (4, pg. 41) if $0 \to A \to B \to D \to 0$ is exact, it induces the exact sequence

$$\mathrm{Ext}^{n-1} (A, C) \to \mathrm{Ext}^n (D, C) \to \mathrm{Ext}^n (B, C) \to \mathrm{Ext}^n (A, C) \to \mathrm{Ext}^{n+1} (D, C) \ldots$$

So we have $0 \to \mathrm{Hom}\ (F, G) \to \mathrm{Hom}\ (F, T(E)) \xrightarrow{g}$ $\mathrm{Hom}\ (F, E) \xrightarrow[f]{f' \to 0}$ $\mathrm{Ext}^1\ (F, G) \to \mathrm{Ext}^1\ (F, T(E)) \to$ $\mathrm{Ext}^1\ (F, E) \to \ldots$ $\mathrm{Ext}^{n-1}\ (F, E) \to \mathrm{Ext}^n\ (F, G) \to$ $\mathrm{Ext}^n\ (F, T(E)) \to \mathrm{Ext}^n\ (F, E) \to \mathrm{Ext}^{n+1}\ (F, G) \to \ldots$ Im g = Ker f, but since $\mathrm{Hom}\ (E, T(E)) \to \mathrm{Hom}\ (F, E) \xrightarrow{f'} 0$ is exact, Im g = Hom (F, E) so Ker f = Hom (F, E) so Im $f = 0$ so we get $0 \to \mathrm{Ext}^1\ (F, G) \to \mathrm{Ext}^1\ (F, T(E)) \to \mathrm{Ext}^1\ (F, E) \to$ $\mathrm{Ext}^2\ (F, G) \to \ldots.$ But $\mathrm{Ext}^2\ (F, G) \cong \mathrm{Ext}^1\ (K, G)$ so we have $0 \to \mathrm{Ext}^1\ (F, G) \to \mathrm{Ext}^1\ (F, T(E)) \to \mathrm{Ext}^1\ (F, E) \to$ $\mathrm{Ext}^1\ (K, G) \to \mathrm{Ext}^1\ (K, T(E)) \to \mathrm{Ext}^1\ (K, E)$. But, $\mathrm{Hom}\ (K, T(E)) \to \mathrm{Hom}\ (K, E) \to 0$ is exact as K torsion-free so we get $0 \to \mathrm{Ext}^1\ (K, G) \to \mathrm{Ext}^1\ (K, T(E)) \to \mathrm{Ext}^1\ (F, E) \to 0$. By induction if $0 \to \mathrm{Ext}^m\ (F, G) \to \mathrm{Ext}^m\ (F, T(E)) \to$ $\mathrm{Ext}^m\ (F, E) \to 0$ is exact for all F torsion-free, and $m \leq n - 1$, then $0 \to \mathrm{Ext}^m\ (K, G) \to \mathrm{Ext}^m\ (K, T(E)) \to$ $\mathrm{Ext}^m\ (K, E) \to 0$ is exact for all $m \leq n - 1$ since K is torsion-free. Now by isomorphism $0 \to \mathrm{Ext}^n\ (F, G) \to$ $\mathrm{Ext}^n\ (F, T(E)) \to \mathrm{Ext}^n\ (F, E) \to 0$ is exact and the theorem is proved for all $n \geq 1$.

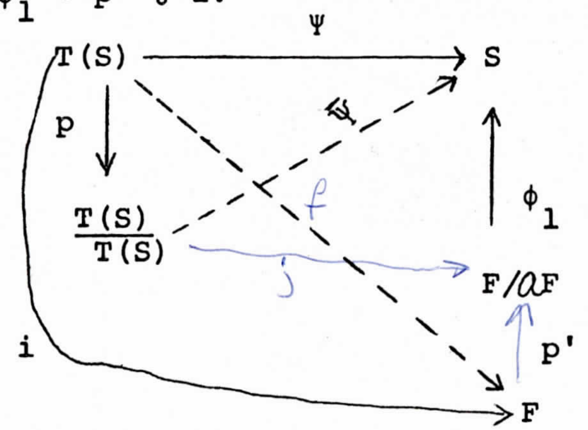
<u>Lemma 1.13</u>: If M and n are A-modules, $M \subset N$, i the connonical injection and $f \circ i$ is an isomorphism where $i\colon M \to N$ $f\colon N \to M$, then $N \cong i(M) \oplus \mathrm{Ker}\ f$.

Proof: $i(m) \cap \mathrm{Ker}\ f = \{0\}$ since if $x \in i(A) \cap \mathrm{Ker}\ f$ then $f(i(x)) = 0$ so $x = 0$ since an isomorphism is an injection.

Now $N = i(M) \oplus \text{Ker } f$, since clearly $\{i(M) \oplus \text{Ker } f\} \subset N$. Now let $x \in N$, then there exists $y \in M$ such that $f(x) = y$ and y is unique. Now $f(i(y)) = f(y) = y$, so $f(x) - f(y) = 0$, and hence $f(x-y) = 0$ which implies that $x-y \in \text{Ker } f$. This means that there exists $z \in \text{Ker } f$ such that $x - y = z$. But since $x = y + z$ and since $y \in i(M)$, $x \in i(M) \oplus \text{Ker } f$. Since if also $x = y + z'$ where $z' \phi \text{ Ker } f$, then $y + z' - (y + z) = 0$. So $z' - z = 0$, so $z' = z$ and x is uniquely represented.

Theorem 1.14: If S is a simple A-module, $\mathfrak{a} \subset A$ is the annihilation of S and Ψ: $T(S) \to S$ is a torsion-free covering of S, then T(S) is a direct summand of any torsion-free module F containing T(S) such that $\mathfrak{a}T(S) = \mathfrak{a}F \cap T(S)$.

Proof: Let F be a torsion-free module containing T(S) such that $\mathfrak{a}T(S) = \mathfrak{a}F \cap T(S)$, $T(S) \xrightarrow{i} F$, i connonical injection induces a map $T(S)/\mathfrak{a}T(S) \xrightarrow{i} F/\mathfrak{a}F$ which is an injection. i.e., j: $T(S)/\mathfrak{a}T(S) \to F/\mathfrak{a}F$ where $j(x + \mathfrak{a}T(S)) = x + \mathfrak{a}F$ is a function. If $x + \mathfrak{a}T(S) = y + \mathfrak{a}T(S)$ then $x - y \in \mathfrak{a}T(S)$ so $x - y \in \mathfrak{a}F$ so $j(x-y + \mathfrak{a}T(S)) = 0 + \mathfrak{a}I = j(x + \mathfrak{a}T(S)) - j(y = \mathfrak{a}T(S))$ so $j(x + \mathfrak{a}T(S)) = j(y + \mathfrak{a}T(S)) = j(y + \mathfrak{a}T(S))$. j is an injection, since suppose $j(x + \mathfrak{a}T(S) = 0 + F$, then $i(x) + \mathfrak{a}F = 0 + \mathfrak{a}F$ so $i(x) = x \in \mathfrak{a}F$, but $x \in T(S)$ so $x \in \mathfrak{a}T(S)$ so $x + \mathfrak{a}T(S) = 0 + \mathfrak{a}T(S)$, so Ker $j = 0$. Since S simple, any $x \in S$, $x \neq 0$, generates S. Pick $x \neq 0$. Define T_x: $A \to S$ by $T_x(a) = ax$, this is a homomorphism. Ker $T_{(x)} = \{a \in A | ax = 0\}$ = annihilation of $S = \mathfrak{a}$. $A/a \simeq S$, so $\mathfrak{a}$ is maximal, so A/a is a field. $F/\mathfrak{a}F$ is an A/a module with the obvious maps and definition of addition so $F/\mathfrak{a}F$ is a vector space over A/a. Let $\{X_\lambda\}$ $\lambda \in L$ be a basis of $F/\mathfrak{a}F$, then $F/\mathfrak{a}F \cong \bigoplus_{\lambda \in L} (A/a)_\lambda$. Define ϕ: $F/\mathfrak{a}F \to S$ as follows: let $\{z_i\}_{i\ I}$ be a basis of $T(S)/\mathfrak{a}T(S)$ as an A/a vector space. Expand $\{j(z_i)\}_{i\in I}$ into $\{x_i\}_{i\in I \cup K}$ a basis of $F/\mathfrak{a}F$ as an A/a vector space. $I \cap K = \phi$. $x_i = j(z_i)$ $i \in I$ and $\phi(x_i) = 0$ if $i \in K$ where $\overline{\Psi}$ is the map induced by Ψ and the connonical map p: $T(S) \to T(S)/\mathfrak{a}T(S)$, i.e., $\overline{\Psi}(x + \mathfrak{a}T(S)) = \Psi(x)$. Note if x $\mathfrak{a}T(S)$, $x = \Sigma a_i y_i$ for some $\{a_i\} \in \mathfrak{a}\{y_i\} \in T(S)$ so $\overline{\Psi}(x + \mathfrak{a}T(S)) = \Psi(x) = \Psi\Sigma(a_i y_i) = \Sigma\Psi(a_i y_i) = \Sigma a_i\ \Psi\ (y_i) = 0$ as $a_i \in \mathfrak{a}$ for each i. Thus letting p and p' denote the connonical mapping from T(S) into $T(S)/\mathfrak{a}T(S)$ and from F into $F/\mathfrak{a}F$ we get $\Psi = \overline{\Psi} \circ p = \phi_1 \circ p' \circ i$ as $\Psi = \phi_1 \circ j \circ p$ as $j \circ p = p' \circ i$, i. e., $j \circ p(x) = j(x + \mathfrak{a}T(S)) = x + \mathfrak{a}F$. $p' \circ i(x) = p'(x) = x + \mathfrak{a}F$. So $\Psi = \phi_1 \circ p' \circ i$.



But since F is torsion-free there exists a linear mapping f: $F \to T(S)$ such that $\Psi \circ f = \phi_1 \circ p'$ so $\Psi \circ f \circ i = \phi_1 \circ p' \circ i = \Psi$, hence $f \circ i$ is an automorphism of T(S) by Theorem 1.11 so that $i(T(S)) = T(S)$ is a direct summand of F by Lemma 1.13.

This completes this section of the paper. So far it has been proven that for A an integral domain and E any A-module, there exists a torsion free covering module of E and a torsion-free covering of E. We would like to investigate the existence of similar objects for rings in general but this is beyond the scope of this paper. However, in order to make such an investigation it would first be necessary to define in a somewhat different light the concepts of torsion-free, divisible, and pure and to study these properties for rings in general. This is what will be done to some extent in the next two sections.

Section 2.

Using the definition of torsion-free modules as given earlier verbatum, we find that the only rings with torsion-free modules are integral domains. That is if $m \varepsilon M$ a right module over a ring R with non-zero elements x, y such that $xy = 0$, then $mx = 0$ or $(mx)y = 0$ so each element of M is torsion for each right R-module M. There does not exist a right torsion-free module over R. Also if we assume that R is not commutative (even though it is an integral domain) we do not know that the torsion elements of M, T(M), for M a right R-module form a submodule.

In this section we shall take the following as our definition of torsion. An element m of a right R-module M is a torsion element if $Md = 0$ for some regular element (some non-zero divisor) d of R. Using this definition, which is the same as the usual one in the case of integral domains, every ring is a torsion-free module over itself. Since if $x \varepsilon R$ and $xd = 0$ for some regular element d then $x = 0$ (otherwise d not regular). So there are no non-zero torsion elements. In a later theorem it will be shown under what conditions torsion submodules exist. We shall also define divisible as follows: An R-module M is divisible if $Md = M$ for every regular element d of R. The condition that d be regular is necessary since otherwise the ring of quotients would not be divisible.

Definition: Let R and S be rings satisfying the following: (1) $R \subset S$, (2) every regular element of R has a two-sided inverse in S (S has identity), (3) every element has the form rd^{-1} for properly chosen r, d in R, with d regular, then we shall call S a right quotient ring of R. If 3 is replaced by $d^{-1} r$ then S is a left quotient ring of R.

Definition: A ring R is said to have the common multiple property (CM) if for every x, d in R with d regular, there exists d_1, y in R with d_1 regular, such that $xd_1 = dy$.

<u>Lemma 2.1</u>: If R satisfies (CM) and $dx = d_1$ where d and d_1 are regular then x is regular.

Proof: Suppose $y \varepsilon R$, $y \neq 0$, $xy = 0$, then $d(xy) = 0$ so $(dx) y = 0$, so $d_1 y = 0$, so d_1 is not regular, a contradiction. Suppose $y \varepsilon R$, $y \neq 0$ and $yx = 0$, then apply (CM) to d, d_1, we obtain i, f, e regular such that $de = d_1 f$, then $(dx) f = de$ so $xf = e$. If $yx = 0$, $y(xf) = (yx) f = ye = 0$, and e is not a regular contradiction.

<u>Corollary 2.1'</u>: If d_1 and d_2 are regular in R and R has (CM) then there exists c_1, c_2 in R such that $d_1 c_1 = d_2 c_2$.

Proof: c_1 is regular but d_1, c_1 regular implies $d_1 c_1$ regular and by lemma 2.1 c_2 is regular so given d_1, d_2 regular there exists c_1, c_2 regular such that $d_1 c_1 = d_2 c_2$.

Theorem 2.2: R has a right quotient ring iff R has regular elements and satisfies (CM).

Proof: If R has a right quotient ring S then $1 \in S$ so $1 = rd^{-1}$ for r, d in R, d regular. So R has regular elements. Now let $x, d \in R$ with d regular, then d has a two-sided inverse in S and $x \in S$ so $d^{-1} x \in S$ so $d^{-1} x = yd_1^{-1}$, for some y, d_1 in R, d_1 regular. So $dd^{-1} x d_1 = dy d_1^{-1} d_1$ so $xd_1 = dy$, so given $x, d \in R$ d, regular, there exists $d_1, y \in R$ d_1 regular such that $xd_1 = dy$; hence, R has (CM). Now if R has regular elements and satisfies (CM), then R has a right ring of quotients for look at R x D. D = regular elements of R and define an equivalence relation $(a, b) \approx (c, d)$, if $ad_1 = cb_1$ where $db_1 = bd_1$, b_1 regular and hence, by lemma d_1 regular. Claim that this is independent of the particular b_1, d_1 which give the (CM) of d and b. For if $db_2 = db_2$, pick e, e_2 regular so that $b_2 e_2 = b_1 e_1$ then $bd_2 e_2 = db_2 e_2 = db_1 e_1 = bd_1 e_1$. Since b is regular, we end up with $d_2 e_2 = d_1 e_1$. From $ad_1 = cb_1$ we get $ad_2 e_2 = ad_1 e_1 = cb_1 e_1 = cb_2 e_2$ and the regularity of e_2 permits us to conclude that $ad_2 = cb_2$. We now see that the relation is an equivalence relation.

reflexive $(a, b) \approx (a, b)$ as $ab = ab$ and $bb = bb$

symetric if $(a, b) \approx (c, d)$ then $ad_1 = cb_1$ where $db_1 = bd_1$ then $cb_1 = ad_1$, where $bd_1 = db_1$ so $(c, d) \approx (a, b)$.

<u>transitive</u> if $(a, b) \approx (c, d)$ and $(c, d) \approx (e, f)$ then $ad_1 = cb_1$ where $db_1 = bd_1$ and $cf_1 = ed_1'$ where $fd_1' = df_1$. Then by letting $f_1' = d_1 g_1$ and $b_1' = d_1' g_2$ where g_1, g_2 such that $b_1 g_1 = f_1 g_2$ we have $af_1' = eb_1'$ where $fb_1' = bf_1'$ since $af_1' = ad_1 g_1 = cb_1 g_1 = cf_1 g_2 = ed_1 g_2 = eb_1'$ and $fb_1' = fd_1' g_2 = df_1 g_2 = db_1 g_1 = bd_1 g_1 = bf_1'$.

We now introduce operations $\bar{+}$, $\bar{\cdot}$, which render R x D a ring. Define $\bar{+}$ by $(a, b) \bar{+} (c, d) = (ad_1 + cb_1, db_1)$ where $db_1 = bd_1$ both b_1, d_1 regular. Define $\bar{\cdot}$ by $(a, b) \bar{\cdot} (c, d) = (ca_1, bg_1)$ where $ag_1 = da_1$ g_1 regular. These operations are well defined, closed, and associative (o, d) is $\bar{+}$ identity. $\bar{+}$ commutes and $\bar{\cdot}$ is distributive over $\bar{+}$ and (d, d) is $\bar{\cdot}$ identity. Now let d be regular in R then $\bar{d} = (dd_1, d_1)$ for any d_1 regular in R. $\bar{I} = (d, d)$ for any regular d in R but $(dd_1, d_1)(d_1, dd_1) = (d_1, d_1) = \bar{I}$ and $(d_1, dd_1) \bar{\cdot} (dd_1, d_1) = (dd_1, dd_1) = \bar{I}$ as dd_1 regular since d, d_1 regular. So d has a two-sided inverse. Let $x \in S$ then $x = (y, d)$ for some $y \in R$, d regular in R. But $(y, d) = (y, d)((dd_1, d_1)(d_1, dd_1))(y, d) = [(y, d)(dd_1, d_1)]$ $(d_1, dd_1) = dd_1 a_1, dg_1$ where $yg_1 = d_1 a_1$ and g_1 regular so dg_1 regular but $(dd_1 a_1, dg_1) \approx (d_1 a_1, g_1) \approx (gg_1, g_1) = \bar{y}$ but $(d_1, dd_1) = (dd_1, d_1)^{-1} = \bar{d}^{-1}$ so $(y, d) = yd^{-1}$ $y \in R$, $d \in R$, d regular and the properties of a right ring of quotients are satisfied by identifying $x \in R$ with (xd, d) for any d regular in R. Clearly this identification is an injective ring homomorphism.

Theorem 2.3: If R is a ring which has a right ring of quotients S and a right ring of quotients S' then S is isomorphic to S'.

Proof: Define $f: S \to S'$ by if $x \in S$, $x = rd^{-1}$ for some $r, d \in R$, d regular, but since $r, d \in R$, d regular, r, d and $d^{-1} \in S'$ so $rd^{-1} \in S'$. So let $f(x) = f(rd^{-1}) = rd^{-1} \in S'$ $x \in S$. f is a function since if $x \approx X'$ then $(r, d) \approx (r', d')$ so $f(x) \approx f(x')$ by the nondependence of the equivalence on the particular regular elements chosen to get the equivalence. If $f(x) = 0$, then $rd^{-1} = 0$ so $x = 0$. f is 1^{-1}. Clearly f is onto $f(x + g) = f(rd^{-1} + sd_o^{-1}) = f(r, d) + (s, d_o)) = f(rd_1 + sb_1, d_o b_1)$ where $d_o b_1 = dd_1 = (rd_1 + sb_1)(d_o b_1)^{-1} = rd_1(d_o b_1)^{-1} + sb_1(d_o b_1)^{-1} = rd_1(dd_1)^{-1} + sb_1(d_o b_1)^{-1} = rd^{-1} + sd_o^{-1} = f(x) + f(y)$.

Lemma 2.4: Let S be a right quotient ring of R. Then

1) For each right ideal J* of S, $J^* = (J^* \cap R)S$

2) If J and K are right ideals of R whose sum is direct, then $(J + K)S = JS + KS$.

Proof:

1) Suppose J* is a right ideal of S. Let $x \in J^*$, then $x = rd^{-1}$ for some r, d in R with d regular as $J^* \subset S$. $xy \in J^*$ for all $y \in S$, $d \in S$ so $xd \in J^*$ so $r \in J^*$ and $r \in R$ so $r \in J^* \cap R$, $d^{-1} \in S$ so rd^{-1} $(J^* \cap R)\ S$ so $x \in (J^* \cap R)S$ so J^* $(J^* \cap R)S$.

Now let $x \in (J^* \cap R)S$, then $x \in J^*S$ and $x \in J^*$ as J^* a right ideal of S, so $(J^* \cap R)S \subset J^*$ so $J^* = (J^* \cap R)S$.

2) Let J and K be right ideals of R shose sum is direct. Clearly, $(J \oplus K)\ S = JS + KS$. Claim $JS + KS$ is direct. That is $JS \cap KS = 0$. If $x \in JS \cap KS$ then $x = yzd^{-1}$, $y \in J$, $z, d \in R$, d regular. And $x = y_1, z_1, d_1^{-1}$, y_1 K, z_1, d_1 R, d_1 regular, $yz = y2$ for some $y_2 \in J$ and $y_1, z_1 = y_3$ for some $y_3 \in K$ so $y_2d^{-1} = y_3, d_1^{-1}$. Now given d, d_1, there exists regular $c, c_1 \in R$ such that $dc = d_1, c_1$ so $y_2\ d^{-1}\ dc = y_3, d_1^{-1}, d_1, c_1$ and $y_2\ c = y_3, c_1$. But $y_2c \in J$ as j a right ideal and $c \in R$ and y_3c_1, K as K a right ideal of R so $y_2c = y_3c_1 = 0$, but since c, c_1 regular $y_2 = 0$ and $y_3 = 0$. So y, z and $y_1, z_1 = 0$, so $x = 0$.

Lemma 2.5: If R has a right quotient ring S, and if $s_i = r_i, d_1^{-1}$, ϵs ($_i = 1, \ldots, n$; $r_i, d_i, \epsilon R$, d_1 regular), then there exists elements x_i, $d \in R$, such that $s_i = x_i\ d^{-1}$.

Proof: If $n = 1$, $s_1 = r_1d_1^{-1}$ and by S a right quotient ring if $n = 2$, there exists c_1, c_2 regular such that $d_1c_1 = d_2c_2$ and $s_1 = r_1c_1(d_1\ c_1)^{-1}$; $s_2 = r_2\ c_2\ (d_2\ c_2)^{-1} = r_2c_2\ (d_1c_1)^{-1}$ $d = d_1c_1$; $x_1 = r_1c_1$; $x_2 = r_2c_2$. Suppose for $i = k$ there exists $(x_i)_{i=1}^{K}$ and d such that $s_i = x_id^{-1}$ and let

s_{K+1} εS, that is, $S_{K+1} = X_{K+1} d^{-1}_{K+1}$, then there exists c, c_{K+1}, regular in R such that $dc = d_{K+1} C_{K+1}$. Then $d' = dc = d_{K+1} c_{K+1}$ and $x_i' = x_i c_{i=1}^{K}$ and $r_{K+1} c_{K+1}$, $s_i (x_i c)(dc)^{-1}$ $i=1, \ldots, K$, $s_{K+1} = r_{K+1} C_{K+1} (d_{K+1} c_{K+1})^{-1} = r_{K+1} c_{K+1} (dc)^{-1}$. And by induction the lemma is proved.

Theorem 2.6: The set of torsion elements of each right R-module forms a submodule iff R has a right quotient ring.

Proof: Suppose R has a right quotient ring. Let M be a right R-module and T the set of torsion elements of M. $O \varepsilon T$ as R has a right quotient ring and therefore has regular element d and od = 0 so 0 torsion. Now if t_1, $t_2 \varepsilon T$, then $t_1 d_1 = t_2 d_2 = 0$ for some d_1, d_2 εR both regular. By Corollary 2.1' there exists c_1, c_2 regular in R such that $d_1 c_1 = d_2 c_2$. Now $(t_1 - t_2)(d_1 c_1) = 0$ so $t_1 - t_2$ εT so T is a subgroup of M. We need only that $tx \varepsilon T$ for each $x \varepsilon R$ with $t \varepsilon T$. If $x \varepsilon R$, $t \varepsilon T$, then td = 0 for some d regular in R. But this implies by (CM) that there exists d_1, y such that $xd_1 = dy$ and $(tx)d_1 = (td)y = 0$ so $tx \varepsilon T$ and T is a submodule. Now suppose the set of torsion elements of each right R-module forms a submodule. Then since ϕ is not a submodule the set of torsion elements of each right R-module is non-empty. Since R is a right R-module the set of torsion elements of R is non-empty and forms a submodule. So since 0 is an element of each module, 0 is a torsion element, so there exists $d \varepsilon R$, d regular such that $0d = 0$. So R has regular elements. Let x, ϕ be in R with d regular. Now dR/d^2R is a right R-module. If $dx + d^2R \varepsilon dR/d^2R$ then $d(-x) + d^2R$ is its inverse. $d0 + d^2R$ is the identity. $(dx + d^2R)y = dxy + d^2R \varepsilon dR/d^2R$ as $xy \varepsilon R$. Then the set of torsion elements of dR/d^2R forms a submodule. Now $d + d^2R$ is a torsion element as $(d + d^2R)d^2 = dd^2 + d^2R = d^2 d + d^2R = 0 + d^2R$ as $d^2d \varepsilon d^2R$. Hence, for any $x \varepsilon R$ $(d + d^2R)x$ is also a torsion element by torsion elements being a submodule. Hence, for some regular d', $(dx + d^2R)d' = dxd' + d^2R = 0 + d^2R$. That is $dxd' \varepsilon d^2R$; hence, $dxd' = d^2y$ for some $y \varepsilon R$. So given x, $d \varepsilon R$, d regular, there exists d', y such that $xd' = dy$ and we have the (CM). So R has regular elements and satisfies (CM) so by Theorem 2.2 R has a right quotient ring.

Proposition 2.7: Let R have a right quotient ring S and let M be a right R-module. Then M is an R-submodule of some S-module iff M is torsion-free. When the condition holds, every element of MS has the form md^{-1} ($m \varepsilon M$, $d \varepsilon R$) and $MS \cong M \otimes_R S$ under the correspondence $MS \to M \otimes S$.

Proof: Suppose M is torsion-free. Then the map $m \to m \otimes 1$ is an R homomorphism from $M \to M \otimes S$ as S is both a right and left R-module so $M \otimes S$ is a right R-module. $M \otimes S$ is a right S-module by the map $(m \otimes s)t = (m \otimes st)$ $st \varepsilon S$. If we get $m \to m \otimes 1$, 1-1 then M would be isomorphic to a submodule of an S-module. Let F be the free abelian group whose generators are the ordered pairs $(m, s) \in M \times S$, and let f be the map of F onto $M \otimes S$ given by $f(\Sigma \pm (m_i, s_i)) = \Sigma \pm m_i \otimes s_i$. Then Ker f is generated by elements of the form $(m_1 + m_2, s) - (m_1, s) - (m_2, s)$, $(m, s_1 + s_2) - (m, s_1) - (m, s_2)$, and $(mr, s) - (m, rs)$ $(r \in R)$. If for some n, $n \otimes 1 = 0$ then $(n, 1) \in$ Ker f so $(n, 1) = \sum_{i=1}^{t} \pm (m_i, s_i)$, where the terms on the right, when properly grouped are among the generators of Ker f (or their negatives). Let d be a common right denominator for the elements s_i (Lemma 2.5), $(n, 1) = \sum_{i=1}^{t} \pm (m_i, x_i d^{-1})$, $f(n, 1) = f(\sum_{i=1}^{t} \pm (m_i, x_i d^{-1})) \doteq n \otimes 1 = \sum_{i=1}^{t} (m_i \otimes x_i d^{-1}) \in M \otimes Rd^{-1}$ so $n \otimes 1 = 0$ in $M \otimes Rd^{-1}$. But $M \otimes Rd^{-1} \cong M \otimes R \cong M$ (as additive groups) under the correspondence $m \otimes rd^{-1} \to m \otimes r \to mr$. Hence, $0 = n \otimes 1 = n \otimes dd^{-1} \to nd$. Since d is invertible in S, and hence regular in R and n not torsion, $n = 0$. Hence, M is contained isomorphically in $M \otimes S$ with the imbedding $m \to m \otimes 1$. Now if M is contained in some S-module and $md = 0$ $(m \in M$, d regular in R) then $0 = mdd^{-1} = m$ so m is torsion-free.

Every element of MS has the form $m_i s_i$. If we write $s_i = r_i d^{-1}$ (Lemma 2.5) then $\Sigma m_i s_i = (m_i r_i) d^{-1}$ which is of the form md^{-1}. Note that this does not imply that any element can be written with the same d but that given an element, such a d can be found for that element. Similarly, every element of $M \otimes S$ is of the form $m \otimes rd^{-1} = mr \otimes d^{-1} = m' \otimes d^{-1}$. Hence by the elementary properties of tensor products, the map $m \otimes s \to ms$ of $M \otimes S$ onto MS is well defined. It is 1-1 since $m \otimes d^{-1} \to 0 = md^{-1}$ implies $m = 0$ and hence $m \otimes d^{-1} = 0$.

Lemma 2.75: Any R linear map between S modules is S-linear.

Proof: Let M and N be S-modules and let f be an R-linear map from M to N. Then $f(m_1 + m_2) = f(m_1) + f(m_2)$ and if $r \varepsilon R$ then $f(rm) = rf(m)$. We only need to show that if $s \varepsilon S$ then $f(sm) = sf(m)$. If $s \varepsilon S$ then $s = rd^{-1}$ for some d regular in R so $f(sm) = f(rd^{-1}m) = rf(d^{-1}m)$ as M is an S-module so $d^{-1}m \varepsilon M$ and f is R-linear. Now $f(d^{-1}m) = d^{-1}df(d^{-1}m) = d^{-1}f(dd^{-1}m) = d^{-1}f(m)$ as $d \varepsilon R$ and M is R-linear so $f(sm) = f(rd^{-1}m) = rd^{-1}(f(m))$ so f is S-linear.

Corollary 2.7': Let R have a right quotient ring S, let M and N be R-submodules of right S-modules, and let f be an R-homomorphism of M into N. Then f^*: $MS \to NS$ defined by $f^*(ms) = f(m)s$ extends f to an S-homomorphism of MS into NS. If f is one-to-one or onto, so is f^*.

Proof: Since if M is a right R-module then $M \otimes S$ is a right S-module by the map $M \otimes S \times S \to M \otimes S$ defined by if $m \otimes r_o d_o^{-1} \varepsilon S$ then $(m \otimes r_o d_o^{-1})(r_1 d_1^{-1}) = m \otimes r_1 r_2 d_o^{-1} d_2^{-1}$ where $r_o d_2 = d_1^{-1} r_2$ and since if f is an R-linear map from $M \to N$ where N is another right R-module then $f \otimes 1_S$ is an S-linear Map (by the preceeding Lemma) from $M \otimes S \to N \otimes S$. The existence of $f^* \cong f \otimes 1s$ is guaranteed as by the theorem $MS \cong M \otimes_R S$ and $NS \cong N \otimes_R S$. Now if f is onto, that is

$M \to N \to 0$ is exact then tensoring we have $M \otimes S \to N \otimes S \to 0$ and $f \otimes 1s$ hence f^* is onto. Now if f is 1-1 then Ker f = 0 so Ker $f^* = 0$ as if $f^*(md^{-1}) = 0$ then $f(m)d^{-1} = 0$ so $f(m) = 0$, $m = 0$ so $md^{-1} = 0$ and f^* is 1-1.

Over a commutative integral domain every injective module is divisible, every torsion-free divisible module is injective. We need to find out what happens for rings in general. (In particular for rings with right rings of quotients)

<u>Theorem 2.8</u>: For R an arbitrary ring with identity, every injective R-module is divisible.

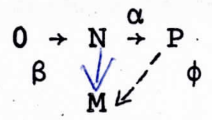
Proof: Let M be an injective right R-module, $m \in M$ and $d \in R$, d regular. The correspondence $dr \to mr$ of dR into M is well defined as d is not a zero divisor. If $dr = dr'$ then $d(r-r') = 0$ so $r - r' = 0$ so $r = r'$. This map is obviously an R-homomorphism and therefore it can be extended to an R-homomorphism ϕ from R to M as M injective. Suppose $\phi(1) = m_1$. Then $m_1 d = \phi(1d) - \phi(d1) = m1 = m$ so if $m \in M$ there exists $m_1 \in M$ such that $m_1 d = m$ so $M \subset Md$ so $Md = M$. ($Md \subset M$ as M an R-module) and M is divisible.

<u>Corollary 2.81</u>: Every module is a submodule of a divisible module.

Proof: Every module is a submodule of an injective module by Lemma 1.7' and by the previous Theorem every injective module is divisible.

<u>Theorem 2.9</u>: Let R have right quotient ring S. The following are equivalent: 1) Every torsion-free divisible right R-module is injective. 2) S is semi-simple. (Here semi-simple is defined as in (4. pg. 12) and is equivalent to every S-module is injective.)

Proof: Suppose 1) and let N be an S-module. Then by 2.7 M is a torsion-free R-module. Now since M is an S, module, for any $m \varepsilon M$, $d \varepsilon R$ d regular $md^{-1} \varepsilon M$ so $m = m_1 d$ for some $m_1 \varepsilon M$ and M = Md so M is divisible. Hence by hypothesis M is R injective. Now let N, P be two S-modules and let α be such that $0 \to N \overset{\alpha}{\to} P$ is exact and let β be such that



Since N and P are S-modules they are R-modules and α and β are R-homomorphisms, so there exists an R-homomorphism ϕ: $P \to M$ such that the diagram commutes since M is R injective. By Lemma 2.75 ϕ is S-linear.

Hence M is injective and since M arbitrary every S-module is injective so S is semi-simple. Suppose 2) that is suppose S is semi-simple. Then every S-module is injective. Let M be a torsion-free divisible R-module and let J be an R ideal and f an R-homomorphism from J to M. Now J is a torsion-free R-module; hence, a submodule of an S-module and so is M by Proposition 2.7. So f can be extended to an S homomorphism f* of $JS \to MS = M$ as M divisible and torsion-free; therefore, an S-module. Then since $JS \overset{\phi}{\to} S$ is injective where ϕ is the natural map and M is injective by hypothesis since it is an S-module, there exists a map f' which makes the following diagram commute.

$$\begin{array}{ccc} JS & \xrightarrow{\phi} & S \\ f^{*}\downarrow & \swarrow f' & \\ M & & \end{array}$$

The restriction of f' to R satisfies the requirements for M to be R injective.

The following theorem is given without proof:

<u>Theorem 2.10</u>: Let R have a two-sided quotient ring S. Then the following are equivalent.

1) Every divisible right R-module is injective.

2) S is semi-simple and R is right hereditary.

<u>Lemma 2.11</u>: Let R have a right quotient ring S. If every finitely generated, torsion-free, right R-module is a submodule of a free module then every finitely generated right S-module is a submodule of a free S-module.

Proof: Let $M = \sum_{i=1}^{n} m_i S$ be a finitely generated S-module. Then M is torsion-free as an R-module, since every S-module is torsion-free as both an R and S module. (Regular elements of both R and S are invertible in S) Let $M_1 = \sum_{i=1}^{n} m_i R$. Since every finitely generated torsion-free right R-module is a submodule of a free module. M_1 is a submodule of a free R-module M_2. Consider M_2 to be a submodule of $M_2 \otimes_R S$ (which by proposition 2.7 then equals $M_2 S$.) Since $R \otimes_R S \simeq S$, and since tensor products preserve direct sums, $M_2 S$ is a free S-module containing $M = M_1 S$.

$M_1 S = M_1 \otimes S = \sum_{i=1}^{n} m_i R \otimes S = \sum_{i=1}^{n} m_i (R \otimes S) = \sum_{i=1}^{n} m_i S = M$

$M_2 \simeq \oplus R_\alpha M_2 \otimes S \simeq \oplus R_\alpha \otimes S \simeq \oplus (R_\alpha \otimes S) \simeq \oplus S_\alpha .$

(TF) represents every finitely generated torsion-free right module is a submodule of a free module.

Theorem 2.12: Let R have a two sided quotient ring S. Then R satisfies (TF) if and only if every finitely generated right S-module is a submodule of a free S-module.

Proof: If R satisfies (TF) then every funitely generated right S-module is a submodule of a free S-module by Lemma 2.11 and if R has two-sided quotient ring then it has a right quotient ring. We need to prove if every finitely generated right S-module is a submodule of a free S-module then R satisfies (TF). Every S-module is torsion-free as both an R and an S-module. So the theorem could read R satisfies (TF) iff S satisfies (TF). So let $M = \sum_{i=1}^{n} m_i R$ be a finitely generated torsion-free right R-module. Consider M to be a submodule of $M \otimes_R S$. (Proposition 2.7) Then $MS = \sum_{i=1}^{n} m_i S$ (as in Lemma 2.11) $\sum_{i=1}^{n} m_i R \otimes S = \sum_{i=1}^{n} m_i (R \otimes_R S) = \sum_{i=1}^{n} m_i S$ is a finitely generated S-module and hence is a submodule of a free S-module by hypothesis. Since each element has finite support each of the m_i's can be written as a combination of a finite number of basis elements of this free module, we can assume that the free module is finitely generated. Suppose the free module is isomorphic to the direct sum $S^{(K)}$ of K copies of S. In the S-isomorphism $MS \to S^{(K)}$ suppose $m_i \to (s_1, s_2, \ldots, s_K)$. Then $M = \sum_{i=1}^{n} m_i R \cong \sum_{i=1}^{n} (s_{1_i}, s_{2_i}, \ldots, s_{K_i}) R$. Let d be a common left denominator for the nk elements s_{j_i}. That is, let $s_{j_i} = d^{-1} r_{j_i}$ with $d, r_{j_i} \in R$ by Lemma 2.5. Then

$$M \cong \sum_{i=1}^{n} (d^{-1} r_{1_i}, \ldots, d^{-1} r_{k_i}) R \cong \sum_{i=1}^{n} (r_{1_i}, \ldots, r_{k_i}) R \subset R^{(K)}$$

a free R-module.

This compleres section 2. In this section it has been shown that the torsion elements of a right R-module form a submodule if and only if R has a right quotient ring. Also if R has a right quotient ring S then S is semi-simple if and only if every torsion-free divisible right R-module is injective. Toward the end of this section it was also shown that if R has a two-sided quotient ring S, R satisfied (TF) if and only if S did.

Section 3.

In the following work yet another torsion theory will be studied with such questions in mind as, is the torsion freeness of a module equivalent to the vanishing of its torsion part, is it possible to divide any module into its torsion-free and torsion parts, and under what conditions is the torsion-free as defined in this section equivalent to the torsion theories of the other two sections.

In the following, let R be a ring with unit 1, and let A be an R-left module on which 1 acts as the identity. If $r(\lambda)$ denotes the right ideal of R consisting of the right annihilator of $\lambda \varepsilon R$, then the subset $r(\lambda)A$ is so to speak a priori torsion with respect to λ.

Definition: A is called torsion-free if, for every $\lambda \varepsilon R$, $\lambda a = 0$ implies $a \varepsilon r(\lambda)A$.

If $l(\lambda)$ denotes the left ideal of left annihilators of λ then we have the following definition.

Definition: A is said to be divisible if for every $\lambda \varepsilon R$ $l(\lambda)\, a = 0$ implies $a \varepsilon \lambda A$. Similarly we could define these for right modules.

Consider the sequence $R \xrightarrow{\lambda} R \xrightarrow{i} R$ where the first arrow is the left multiplication by λ, the second the connonical injection. Tensoring with A over R yields $A \xrightarrow{i \otimes 1} \lambda R \otimes A \xrightarrow{i \otimes 1} A$ where $\lambda \otimes 1$ is an epimorphism with Kernel $r(\lambda)A$ since $0 \to r(\lambda) \xrightarrow{i} R \to \lambda R \to 0$ is exact so

$r(\lambda) \otimes A \to R \otimes A \overset{\lambda x|}{\to} \lambda R \otimes A \to 0$ is exact so Ker $\lambda x|_A$ = Im $r(\lambda) \otimes A$ in $R \otimes A = A = r(\lambda) A$. The composed map $(i \otimes 1)(\lambda \otimes 1)$ is identified with left multiplication by λ in A.

$Tor_1(R/\lambda R, A)$ is the Kernel of $i \otimes 1A$ since $0 \to \lambda R \to R \to R/\lambda r \to 0$ is exact and from the properties of tor we get the following exact $\to Tor_1(\lambda R, A) \to Tor_1(R, A) \to Tor_1(R/\lambda R, A) \to \lambda R \otimes A \to R \otimes A \to R/\lambda R \otimes A$ and $Tor_1(R, A) \simeq 0$ as R projective and $R \otimes A \simeq A$ so we have the following exact $0 \to Tor_1(R/\lambda R, A) \overset{f}{\to} \lambda R \otimes A \overset{ix\lambda A}{\to} A$ so $Tor_1(R/\lambda R, A) \approx$ Im f = Ker $i \otimes 1A$ from the exactness.

So $Tor_1(R/\lambda R, A) \approx$ Ker $i \otimes 1A$. Since $A \overset{\lambda\otimes|}{\to} \lambda R \otimes A \overset{i\otimes|}{\to} A$ where $\lambda\otimes|$ is a surjection. A/Ker $\lambda\otimes 1 \approx \lambda R \otimes A$ so $A/r(\lambda)A \approx \lambda R \otimes A$ so $Tor_1(R/\lambda R, A)$ = Ker $i \otimes 1A \subset A/r(\lambda)A$.
Ker $i \otimes /A = \{a\varepsilon A | \lambda a = 0\}/r(\lambda)A$ as

$$a + r(\lambda) A \to \lambda \times a \to \lambda a$$

$A/r(\lambda) A \to \lambda R \otimes A \to A$

$\lambda(a + r(\lambda)A) = 0$ iff $\lambda a = 0$ so Tor $(R/\lambda R, A) \approx \{a\varepsilon A | \lambda a = 0\}/r(\lambda)A$ and hence

<u>Proposition 3.1</u>: A is torsion free iff $Tor_1(R/\lambda R, A) = 0$ for every $\lambda \varepsilon R$.

Proof: If $Tor_1(R/\lambda R, A) = 0$ for every $\lambda\varepsilon R$ then $\lambda \varepsilon R \Rightarrow \{a\varepsilon A|\lambda a = 0\}/r(\lambda) A = 0$ so if $a\varepsilon A$ such that $\lambda a = 0$ then $a \varepsilon r(\lambda)A$ and by definition A is torsion-free. Now if A is torsion-free then $\{a\varepsilon A|\lambda a = 0\}/r(\lambda) A = 0$ as torsion-free implies for each $\lambda\varepsilon R$ we have $a \varepsilon r(\lambda)A$ whenever $\lambda a = 0$. So $Tor_1(R/\lambda R, A) = \{a\varepsilon A|\lambda a = 0\}/r(\lambda) A = 0$. Also

$Ext^1(R/R\lambda, A) \approx \{a\varepsilon A | l(\lambda)a = 0\}/\lambda A$. $0 \to R\lambda \to R \to R/R\lambda \to 0$ is exact and from the properties of Ext we have $0 \to Hom(R/R\lambda, A) \to Hom(R, A) \to Hom(R\lambda, A) \to Ext^1(R/R\lambda, A) \to Ext^1(R, A) \to \ldots$. But $Hom(R, A) \approx A$ and $Ext^1(R, A) = 0$ as R projective so we have $0 \to Hom(R/R\lambda, A) \to A \xrightarrow{f} Hom(R\lambda, A) \xrightarrow{g} Ext^1(R/R\lambda, A) \to 0$ exact. So $Ext^1(R/R\lambda, A) \approx \dfrac{Hom(R\lambda, A)}{Ker\ g} = \dfrac{Hom(R\lambda, A)}{Im\ f}$.

But $0 \to l(\lambda) \to R \to R\lambda \to 0$ is exact, so we get $0 \to Hom(R\lambda, A) \xrightarrow{f'} Hom(R, A) \xrightarrow{g'} Hom(l(\lambda), A) \to \ldots$. So $Hom(R\lambda, A) \approx Im\ f' = Ker\ g' = \{a\varepsilon A | l(\lambda)\ a = 0\}$ as $Hom(R, A) \approx A$. If $f \in Hom(R, A)$ then $f(1) = a_f$ and f is determined by a_f, i.e., $f(x) = xa_f$ $g'(f)$ just restricts f to $l(\lambda)$. That is $g'(f)(x) = xa_f$, $x \in l(\lambda)$ so $Ker\ g' = \{f \in Hom(R, A) | f$ restricted to $l(\lambda) = 0\} = \{a_f \in A | xa_f = 0$ for all $x \in l(\lambda)\} = \{a\varepsilon A | l(\lambda)a = 0\}$.

Now Im f in the sequence $0 \to Hom(R/R\lambda, A) \xrightarrow{f} Hom(R\lambda, A) \xrightarrow{g} Ext^1(R/R\lambda, A) \to 0 = \lambda A$ as, for $a\varepsilon A$ $f(a) = g \in Hom(R\lambda, A)$ such that $g(r\lambda) = (r\lambda)a = r(\lambda a)$ for all $r\varepsilon R$. Now f(a) can be identified with λa in the same manner as Hom (R, A) can be identified with A. So $Ext^1(R/R\lambda, A) \approx \dfrac{Hom(R\lambda, A)}{Ker\ g} \approx \dfrac{Hom(R/R\lambda, A)}{Im\ f} \approx \{a\varepsilon A | l(\lambda)a = 0\}/\lambda A$ and hence

Proposition 3.1': A is divisible iff $Ext^1(R/R\lambda, A) = 0$ for every $x\varepsilon R$.

Definition: A left R module M is said to be flat if whenever K: $A \to B$ is injective then $K \otimes 1: A \otimes M \to B \otimes M$ is injective.

Proposition 3.2: 1) A flat module A is always torsion-free; 2) An injective module A is always divisible; 3) If every left ideal of R is principal, then a divisible module A is injective.

Proof 1): Let A be a flat R-module, then $0 \to \mathrm{Tor}_1(R/\lambda R, A) \to \lambda R \otimes A \overset{i \otimes 1_A}{\to} A$ is exact but $0 \to \lambda R \overset{i}{\to} R$ is exact so $0 \to \lambda R \otimes A \overset{i \otimes 1_A}{\to} R \otimes A \simeq A$ is exact so $\mathrm{Ker}\ i \otimes 1_A = 0$ and since $\mathrm{Tor}_1(R/\lambda R, A) \approx \mathrm{Ker}\ i \otimes 1_A$, $\mathrm{Tor}_1(R/\lambda R, A) = 0$ and by Proposition 3.1, A is torsion-free.

Comment: Clearly if projective then flat, so projective is torsion-free. 2) Let A be an injective module, then $\mathrm{Ext}^1(R/R\lambda, A) = 0$ since $0 \to R\lambda \to R \to R/R\lambda \to 0$ is exact so $0 \to \mathrm{Hom}(R/R\lambda, A) \to \mathrm{Hom}(R, A) \overset{j'}{\to} \mathrm{Hom}(R\lambda, A) \to \mathrm{Ext}^1(R/R\lambda, A) \to \mathrm{Ext}^1(R, A) = 0$. But Hom (..., A) A injective is an exact contravariant functor (4, page 39) so $0 \to \mathrm{Hom}(R/R\lambda, A) \to \mathrm{Hom}(R, A) \overset{j'}{\to} \mathrm{Hom}(R\lambda, A) \to 0$ so j' is a surjection and we have $0 \to \mathrm{Ext}^1(R/R\lambda, A) \to 0$ exact and so $\mathrm{Ext}^1(R/R\lambda, A) = 0$ and A is divisible by Proposition 3.1'. 3) Let A be a divisible R-module. Then if for each left ideal L and each map $f \in \mathrm{Hom}(L, A)$ there exists a map $f' \in \mathrm{Hom}(R, A)$ such that f' is an extension of f then A is injective. Let L be a left ideal. Then we have the exact sequence $0 \to L \overset{i}{\to} R \overset{\phi}{\to} R/L \to 0$. Hom across with A we get $0 \to \mathrm{Hom}(R/L, A) \to \mathrm{Hom}(R, A) \overset{i'}{\to} \mathrm{Hom}(L, A) \to \mathrm{Ext}^1(R/L, A) = 0$. We have the 0 on the right because L is principal and is therefore $R\lambda$ for some $\lambda \varepsilon R$ and A is divisible. Hence Hom (R, A) $\overset{i'}{\rightarrow}$ Hom (L, A) $\rightarrow$ 0 and we get i' a surjection. That is, for each f ε Hom (L, A) there exists f' ε Hom (R, A) such that f' is an extension of f so A is injective.

Definition: An extension of R-modules (*) $0 \rightarrow A \overset{\alpha}{\rightarrow} \beta \rightarrow C \rightarrow 0$ (exact) is said to be pure if it has one of the two following equivalent properties.

1) A B = λA for every $\lambda \varepsilon$R.

2) If λC = 0 for some cεC, then there exists bεB such that β(b) = C and λb = 0.

(in (1) A is identified with αA B.) These are equivalent respectively to

1′) R/λR $\otimes$ A $\rightarrow$ R/λR $\otimes$ B is a monomorphism for every $\lambda \varepsilon$R.

2′) Hom (R/Rλ, B) $\rightarrow$ Hom (R/Rλ, C) is an epimorphism for every $\lambda \varepsilon$R.

Proof: 1) $\rightarrow$ 2) Suppose A λB = A for all $\lambda \varepsilon$R. If λb ε Ker β = A, λb = $\alpha(\lambda a)$ for some aεA, so if λc = 0 then pick bεB such that β(b) = c. Now $\beta(\lambda b) = \lambda\beta(b) = \lambda c = 0$, so $\lambda b = \alpha(\lambda a)$ for some aεA. Consider $\lambda(b - \alpha(a)) = \lambda b - \lambda(\alpha a) = \lambda b - \alpha(\lambda a) = \lambda b - \lambda b = 0$, $\beta(b - \alpha(a)) = \beta(b) - \beta(\alpha(a)) = c - 0 = c$ as A = Ker β. So there exists b_o ε B such that $\lambda(b_o) = 0$ and $\beta(b_o) = c$ namely $(b - \alpha(a))$.

2) $\rightarrow$ 1): Suppose λC = 0 for cεC implies there exists some bεB such that b = 0 and β(b) = c. Clearly λA A λB as A a submodule of B. Let α(a) $\varepsilon\lambda$B, that is $\alpha(a) = \lambda b$ for

some $b \in B$, then $\beta(\alpha(a)) = \beta(\lambda b) = \lambda\beta(b) = \lambda c = 0$ for some c in C. So by hypothesis there exists $b' \in B$ such that $\lambda b' = 0$ and $\beta(b') = c$. Now $(b-b') \in \operatorname{Ker} \beta$ as $\beta(b-b') = \beta(b) - \beta(b') = c - c = 0$ so $b - b' \in A$. So $\lambda(b-b') \in \lambda A$ but $\lambda(b-b') = \lambda b - \lambda b' = \lambda b$ as $\lambda b' = 0$ so $\lambda b \in \lambda A$ and $A \quad \lambda B = \lambda A$ so $A \quad \lambda B = \lambda A$.

1') $\rightarrow$ 1): If $R/\lambda R \otimes A \rightarrow R/\lambda R \otimes B$ is a monomorphism for every $\lambda \in R$, then since $R/\lambda R \otimes A \approx A/\lambda A$ as $0 \rightarrow \lambda R \overset{i}{\rightarrow} R \overset{\phi}{\rightarrow} R/\lambda R \rightarrow 0$ exact gives $\lambda R \otimes A \overset{i\otimes 1}{\rightarrow} R \otimes A \approx A \overset{\phi\otimes 1}{\rightarrow} R/\lambda R \otimes A \rightarrow 0$ exact where i is the connonical injection, ϕ the connonical surjection. So $R/\lambda R \otimes A \approx A/\ker \phi\otimes 1 = A/\operatorname{Im} i\otimes 1$. $\operatorname{Im} i\otimes 1 = \lambda A$ as $i\otimes 1$: $\lambda R \otimes A \rightarrow A$ by $i\otimes 1\ (\lambda r \otimes a) = \lambda ra = \lambda a'$, $a' = ra$, that is $\operatorname{Im}(i\otimes 1) = \lambda R \otimes A \approx \lambda RA = \lambda A$ as $\lambda RA \subset \lambda A$ and R having unit gives $\lambda RA = \lambda A$. So $R/\lambda R \otimes A \approx A/\lambda A$ and similarly $R/\lambda R \otimes B \approx B/\lambda B$. So $0 \rightarrow A/\lambda A \rightarrow B/\lambda B$ is exact for each $\lambda \in R$. So if $a \in A$, $a \in \lambda B$ then $a + \lambda B = 0$. So since $a + \lambda b$ is image of $a + \lambda A$, $a + \lambda A$ must be 0 also, so $a \in \lambda A$ so $A \cap \lambda B = \lambda A$ for all $\lambda \in R$ as $\lambda A \subset A \cap \lambda B$ clearly.

1) $\rightarrow$ 1'): If $A \cap \lambda B = \lambda A$ for every $\lambda \in R$ then $A/\lambda A \overset{i'}{\rightarrow} B/\lambda B$ is an injection for every $\lambda \in R$ as if $i'(a + \lambda A) = 0$ in $B/\lambda B$ then $a \in \lambda B$ hence $a \in \lambda A$ so $a + \lambda A = 0$ in $A/\lambda A$ and $\operatorname{Ker} i' = 0$. So i' injective. Now since $A/\lambda A \approx R/\lambda R \otimes A$ and $B/\lambda B \approx R/\lambda R \otimes B$, $R/\lambda R \otimes A \overset{1\otimes i}{\rightarrow} R/\lambda R \otimes B$ is an injection for each $\lambda \in R$.

2') → 2): Suppose Hom (R/Rλ, B) → Hom (R/Rλ, C) is an epimorphism for every λεR. Now since if f ε Hom (R/Rλ, B) then f is defined by what it does to (1 + Rλ). That is if rεR f(R + Rλ) = rf(1 + Rλ) = rb where b = f(1 + rλ). But 0 = f(λ + R) = λf(1 + Rλ) = λB so if f ε Hom (R/Rλ, B) then f is identified with some bεB such that λb = 0 and we have Hom (R/Rλ, B) ≈ {bεB|λb = 0} and similarly for Hom (R/Rλ, C). So we get for λεR $\{b \varepsilon B | \lambda b = 0\} \xrightarrow{\beta'} \{c \varepsilon C | \lambda c = 0\}$ is a surjection, that is, for each cεC such that λc = 0 there exists bεB such that λb = 0 and β'(b) = c where β' = β restricted to {bεB|λb = 0}.

2) → 2'): Suppose if λC = 0 cεC then there exists bεB such that β(b) = c and λb = 0, then $\{b \varepsilon B | \lambda b = 0\} \xrightarrow{\beta'} \{c \varepsilon C | \lambda c = 0\}$ where β' is β restricted to {bεB|λb = 0} is a surjection for each λεR. But since {bεB|λb = 0} ≈ Hom (R/Rλ, B) and {cεC|λC = 0} ≈ Hom (R/Rλ, C) then for each λεR Hom (R/Rλ, B) → Hom (R/Rλ, C) is a surjection.

<u>Proposition 3.3</u>:

1) C is torsion-free if and only if every extension (*) with C as the factor module is pure.

2) A is divisible if and only if every extension (*) with A as the Kernel is pure.

Proof 1): For any extension (*) 0 → A → B → C → 0 exact gives $Tor_1(R/\lambda R, A) \to Tor_1(R/\lambda R, B) \to Tor_1(R/\lambda R, C) \to R/\lambda R \otimes A \to R/\lambda R \otimes B \to R/\lambda R \otimes C \to 0$ exact. If C torsion-free then $Tor_1(R/\lambda R, C) = 0$ for all λεR so we have $0 \to R/\lambda R \otimes A \to R/\lambda R \otimes B$ for all λεR and (*) is pure by 1'.

If (*) is pure for every extension (*) then we have

Tor_1 (R/λR, B) → Tor_1 (R/λR, C) → R/ R ⊗ A → R/λR ⊗ B

but since (*) is pure we have Tor_1 (R/λR, B) →

Tor_1 (R/λR, C) → 0 and if we take B projective as is permitted by reference 1 (pg. 7) we have Tor_1 (R/λR, B) = 0 so we get Tor_1 (R/λR, C) = 0 and by Proposition 3.1 C is torsion-free.

Proof 2): If A is divisible then Ext^1 (R/Rλ, A) = 0 for all λεR so for any extension (*) 0 → A → B → C → 0 exact gives 0 → Hom R/Rλ, A) → Hom (R/Rλ, B) → Hom (r/R , C)→ Ext^1 (R/Rλ, A) exact. But since A divisible Ext^1 (R/Rλ, A) = 0 so we have Hom (R/Rλ, B) → Hom (R/Rλ, C) → 0 for all λεR and by 2' (*) is pure. If for any extension (*) 0 → A → B → C → 0 exact (*) is pure then Hom (R/Rλ, B) → Hom (R/Rλ, C) → 0 is exact and 0 → Hom (R/Rλ, A) → Hom (R/Rλ, B) → Hom (R/Rλ, C) → Ext^1 (R/Rλ, A) → Ext^1 (R/Rλ, B) → . . . is exact. But these two gives us 0 → Ext^1 (R/Rλ, A) → Ext^1 (R/Rλ, B) and taking B to be injective (1, pg. 9) we have Ext^1 (R/Rλ, B) = 0 so Ext^1 (R/Rλ, A) 0 and A is divisible by Proposition 3.1'.

As a corollary we have the equivalence of the following three statements:

a) Every extension (*) is pure.

b) Every module is torsion-free.

c) Every module is divisible.

Now we will look at submodules, factor modules, etc. of torsion-free or divisible modules.

Proposition 3.5:

1) An estension of a torsion-free module by a torsion-free module yields always a torsion-free module.

2) Also an extension of a divisible module by a divisible module yields a divisible module.

Proof 1): Suppose we have $0 \to A \to B \to C \to 0$ exact with A and C torsion-free. Then we get $\to \mathrm{Tor}_1 (R/\lambda R, A) \to \mathrm{Tor}_1 (R/\lambda R, B) \to \mathrm{Tor}_1 (R/\lambda R, C) \to R/\lambda R \otimes A \to R/\ R \otimes B \to R/\lambda R \otimes C \to 0$ exact for all $\lambda \varepsilon R$ but since A and C torsion-free $\mathrm{Tor}_1 (R/\lambda R, A) = \mathrm{Tor}_1 (R/\lambda R, C) = 0$ for every $\lambda \varepsilon R$. This gives $\mathrm{Tor}_1 (R/\lambda R, B) = 0$ for all $\lambda \varepsilon R$ so B is torsion-free.

Proof 2): Suppose we have $0 \to A \to B \to C \to 0$ exact with A and C divisible. Then we have $0 \to \mathrm{Hom} (R/R\lambda, A) \to \mathrm{Hom} (R/R\lambda, B) \to \mathrm{Hom} (R/R\lambda, C) \to \mathrm{Ext}^1 (R/R\lambda, A) \to \mathrm{Ext}^1 (R/R\lambda, B) \to \mathrm{Ext}^1 (R/R\lambda, C) \to \ldots$ exact. But $\mathrm{Ext}^1 (R/R\lambda, A) = \mathrm{Ext}^1 (R/R\lambda, C) = 0$ for all $\lambda \varepsilon R$ as A and C divisible so we have $\mathrm{Ext}^1 (R/R\lambda, B) = 0$ for every $\lambda \varepsilon R$ and B is therefore divisible.

If C is a factor module of B, we have an exact sequence of the type (*) with α the connonical injection B the connonical surjection and A the kernel of B. $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$. Such a sequence will be called "associated" with $B \to C$. Similarly, if A is a submodule of B, we have an exact sequence

of the type (*) with α the connonical injection, the connonical surjection and C the cokernel of α.
$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$. Such a sequence will be called "associated" with $A \to B$.

Proposition 3.6:

1) A factor module C of a torsion-free module B is torsion-free if and only if the associated exact sequence (*) is pure.

2) Similarly, a submodule A of a divisible module B is divisible if and only if the associated sequence (*) is pure.

Proof 1): If C torsion-free the associated sequence is pure by Proposition 3.3. If the associated sequence $0 \to A \to B \to C \to 0$ is pure then we have for every $\lambda \varepsilon R$
$Tor_1 (R/\lambda R, A) \to Tor_1 (R/\lambda R, B) \to Tor_1 (R/\lambda R, C) \to$
$R/\lambda R \otimes A \to R/\lambda R \otimes B \to R/\lambda R \otimes C \to 0$ exact. Now $Tor_1 (R/\lambda R, B) = 0$ for every $\lambda \varepsilon R$ as B torsion-free and using 1' in the definition of pure we get $Tor_1 (R/\lambda R, C) = 0$ for every $\lambda \varepsilon R$ and hence C is torsion-free.

Proof 2): If A is divisible then by Proposition 3.3 the associated sequence is pure. If the associated sequence is pure and B is divisible then we get for each $\lambda \varepsilon R$
$0 \to Hom (R/R\lambda, A) \to Hom (R/R\lambda, B) \to Hom R/R\lambda, C) \to Ext^1 (R/R\lambda, A) \to$
$Ext^1 (R/R\lambda, B) \to Ext^1 (R/R , C) \to \ldots$ exact. But
$Ext^1 (R/R\lambda, B) = 0$ for every $\lambda \varepsilon R$ as B divisible and using 2' in the definition of pure we get $Ext^1 (R/R\lambda, A) = 0$ for every $\lambda \varepsilon R$ hence A is divisible.

We shall call a ring R a left PP [respectively PF] ring if every principal left ideal of R is projective [respectively flat]. A right PP [respectively PF] ring is defined similarly.

<u>Lemma 3.7</u>: A PP ring is a PF ring.

Proof: Let R be a left PP ring. Then every principal left ideal of R is projective. Let M be a principal left ideal of R and $0 \to A \to B \to B/A \quad 0$ exact for modules A and B. Then we get $\to \mathrm{Tor}_1 (A, M) \to \mathrm{Tor}_1 (B, M) \to \mathrm{Tor}_1 (B/A, M) \to A \otimes M \to B \otimes M$ exact. But $\mathrm{Tor}_1 (B/A, M) = 0$ as M projective by comment in proof Proposition 3.2, so we get $0 \to A \otimes M \to B \otimes M$ exact and M is flat.

<u>Lemma 3.7'</u>: $\mathrm{Ext}^1 (P, A) = 0$ for all A iff P is projective. If P projective $\mathrm{Ext}^1 (P, A) = 0$. Now if $\mathrm{Ext}^1 (P, A) = 0$ for every A, look at $0 \to F \to G \to H \to 0$ exact. Then get $0 \to \mathrm{Hom} (P, F) \to \mathrm{Hom} (P, G) \to \mathrm{Hom} (P, H) \to \mathrm{Ext}^1 (P, F) = 0$. So ϕ a surjection, so P projective.

<u>Lemma 3.8</u>: $\mathrm{Tor}_1 (\lambda R, C) = 0$ for all C if and only if λR is flat.

Proof: If λR is flat and given C, then C can be put into the exact sequence $0 \to K \to P \to C \to 0$ with P projective and we have $\mathrm{Tor}_1 (\lambda R, P) \to \mathrm{Tor}_1 (\lambda R, C) \to \lambda R \otimes K \overset{\phi}{\to} \lambda R \otimes P$. But $\mathrm{Tor}_1 (\lambda R, P) = 0$ as P projective. Kernel $\phi = 0$ as λR flat. So we have $0 \to \mathrm{Tor}_1 (\lambda R, C) \to 0$, that is $\mathrm{Tor}_1 (\lambda R, C) = 0$.

Now suppose Tor_1 $(\lambda R, C) = 0$ for all C and let $0 \to B \to C \to C/B \to 0$ be exact. Then we have Tor_1 $(\lambda R, C/B) \to \lambda R \otimes B \to \lambda R \otimes C$. But by hypothesis Tor_1 $(\lambda R, C/B) = 0$. So λR is flat.

<u>Proposition 3.7</u>: 1) In order that any submodule of a torsion-free left module be again torsion-free, it is necessary and sufficient that R be a right PF ring.

2) In order that any factor module of a divisible left module be again divisible, it is necessary and sufficient that R be a left PP ring.

Proof 1): Let B be torsion-free, and let A be a submodule of B and suppose that R is a right PF ring. Then every principal right ideal of R is flat. Looking at the associated sequence $0 \to A \to B \to C \to 0$ we get $\to Tor_2$ $(R/\lambda R, C) \to Tor_1$ $(R/\lambda R, A) \to Tor_1$ $(R/\lambda R, B) \to$; but since B torsion-free, Tor_1 $(R/\lambda R, B) = 0$. Also we have $0 \to \lambda R \to R/\lambda R \to 0$ so we get Tor_2 $(R, C) \to Tor_2$ $(R/\lambda R, C) \to Tor_1$ $(\lambda R, C) \to Tor_1$ $(R, C) \to$ exact. Now Tor_2 $(R, C) = Tor_1$ $(R, C) = 0$ as R projective and so we have $0 \to Tor_2$ $(R/ R, C) \to Tor_1$ $(\lambda R, C) \to 0$.

λR is a principal right ideal and hence by hypothesis is flat and so by the preceeding lemma Tor_1 $(\lambda R, C) = 0$ and hence Tor_2 $(R/\lambda R, C) \to Tor_1$ $(\lambda R, C) = 0$. Now this gives $0 \to Tor_1$ $(R/\lambda R, A) \to 0$ hence Tor_1 $(R/\lambda R, A) = 0$ and since R is flat for each $\lambda \varepsilon R$, Tor_1 $(R/\lambda R, A) = 0$ for each $\lambda \varepsilon R$, hence A is torsion-free.

Now suppose any submodule of a torsion-free module is torsion-free and let λR be a principal right ideal of R. Also let C be any module. By the lemma we want $Tor_1(\lambda R, C) = 0$. Now given C we can imbed C in an exact sequence $0 \to A \to P \to C \to 0$, with P projective. P is therefore torsion-free ($Tor_1(R/\lambda R, P) = 0$ for all $\lambda \varepsilon R$) hence by hypothesis A is torsion-free and we get the exact sequence $Tor_2(R/\lambda R, P) \to Tor_2(R/\lambda R, C) \to Tor_1(R/\lambda R, A) \to Tor_1(R/\lambda R, P)$. But both ends are 0 as P projective so $Tor_2(R/\lambda R, C) \approx Tor_1(R/\lambda R, A) = 0$ as A torsion-free. Also we have $0 \to \lambda R \to R \to R/\ R \to 0$ exact so we have $Tor_2(R/\lambda R, C) \to Tor_1(\lambda R, C) \to Tor_1(R, C) = 0$ as R projective so $Tor_1(\lambda R, C) = 0$ Since C was arbitrary λR is flat and since λ was arbitrary any principal right ideal of R is flat and hence R is a PF ring.

Proof 2): Let C be a factor module of a divisible module B and suppose R is a left PP ring. Then every principal left ideal is projective. Looking at the associated exact sequence $0 \to A \to B \to C \to 0$, we get $Ext^1(R/R\lambda, B) \to Ext^1(R/R\lambda, C) \to Ext^2(R/R\lambda, A)$, but $Ext^1(R/R\lambda, B) = 0$ as B divisible. Also considering $0 \to R \to \lambda R \to R/R\lambda \to 0$ we get $Ext^1(R, A) \to Ext^1(R\lambda, A) \to Ext^2(R/R\lambda, A) \to Ext^2(R, A)$ exact. But both ends are 0 as R projective so we get $0 = Ext^1(R\lambda, A) \approx Ext^2(R/R\lambda, A)$ as $R\lambda$ is a principal left ideal and therefore projective. Now we get $0 \to Ext^1(R/R\lambda, C) \to 0$. So $Ext^1(R/R\lambda, C) = 0$ so C is divisible.

Now suppose every factor module of a divisible left module is divisible. Let $R\lambda$ be a principal left ideal and let A be any left R-module. By Lemma 3.7' we want $Ext^1 (R\lambda, A) = 0$. Now A can be imbedded in a sequence $0 \to A \to B \to C \to 0$ with B injective, so B is divisible (Proposition 3.2, 2) and by hypothesis so is C. Now we have the exact sequence $Ext^1 (R, A) \to Ext^1 (R , A) \to Ext^2 (R/R\lambda, A) \to Ext^2 (R, A)$ with both ends 0 as R projective so $Ext^1 (R\lambda, A) \approx Ext^2 (R/R\lambda, A) = 0$ as we have the exact sequence $0 = Ext^1 (R/R\lambda, C) \to Ext^2 (R/R\lambda, A) \to Ext^2 (R/R\lambda, B) = 0$. The left side is 0 as C divisible and the right side is 0 as B injective (4, pg. 50). Now since A arbitrary $Ext^1 (R\lambda, A) = 0$ for every A hence R is projective and since λ arbitrary, each principal left ideal is projective hence R is a left PP ring.

<u>Proposition 3.8</u>: 1) A direct sum of torsion-free modules is torsion-free.

2) In order that a direct product of torsion-free modules be always torsion-free, it is necessary and sufficient that for every $\lambda \varepsilon R$, the right annihilator $r(\lambda)$ be finitely generated.

Proof 1): Let $\{A_\gamma\}\gamma\varepsilon\Gamma$ be a collection of torsion-free modules. Let $A = \sum_{\gamma\varepsilon\Gamma} A\gamma$. $Tor_1 (R/\lambda R, A) = Tor_1 (R/\lambda R, A) = \sum_{\gamma\varepsilon\Gamma} 0 = 0$ as (1 pg. 107, Prop. 1.2a) Tor commutes with direct sums and from the property of torsion-free modules $Tor_1 (R/\lambda R, A) = 0$ for each $\lambda \varepsilon R$ if A is torsion-free.

Proof 2): Let $\{A_\gamma\}\gamma\varepsilon\Gamma$ be a collection of torsion-free modules, and let $A = \prod_{\gamma\varepsilon\Gamma} A$. Suppose for every $\lambda\varepsilon R$, $r(\lambda)$ is finitely generated and let $a\varepsilon A$ such that $a = 0$. This implies $\lambda a_\gamma = 0$ for every $\gamma\varepsilon\Gamma$, that is $a = (a\gamma)_{\gamma\varepsilon\Gamma} \varepsilon A$, $a_\gamma \varepsilon A_\gamma$. Now by hypothesis $r(\lambda)$ is generated by a finite number of elements, say $u_1,\ldots,u_r$. Now since A torsion-free for each γ and since $\lambda a_\gamma = 0$, $a_\gamma \; \varepsilon \; r(\lambda)A$, so there exists $a_{i\gamma}$ $i\varepsilon I$, $a_{i\gamma}\varepsilon A_\gamma$ such that $a_\gamma = \sum_{i\varepsilon I} u_i a_{i\gamma}$. Letting $a_i = (a_{i\gamma})_{\gamma\varepsilon\Gamma} \; \varepsilon \; A$ we have $\sum_{i\varepsilon I} u_i a_i = \sum_{i\varepsilon I} u_i (a_{i\gamma}) = (\Sigma u_i a_{i\gamma}) = (a_\gamma) = a$. But $\Sigma u_i a_i \; \varepsilon \; r(\lambda)A$ so $a \; \varepsilon \; r(\lambda)A$ so A is torsion-free.

To prove the converse suppose that a direct product of torsion-free modules is always torsion-free. Now for $\lambda\varepsilon R$ take the direct product $A\lambda = \prod_{\alpha\varepsilon r(\lambda)} R_\alpha$ of isomorphic copies R_α of R over the index set $r(\lambda)$. Let a_λ be the "diagonal" element of A_λ having αth component α for every $\alpha \; \varepsilon \; r(\lambda)$. Then $\lambda a_\lambda = 0$, so since A_λ is torsion-free, that is, the direct product of torsion-free modules $R_\alpha = R$ which by hypothesis is always torsion-free, $a_\lambda \; \varepsilon \; r(\lambda)A_\lambda$, that is, $a_\lambda = \Sigma u_i a_i$ for some finite number of elements u_i of $r(\lambda)$, and $a_i = (a_{i\alpha})$ of $A\lambda$ $(i = 1 \ldots r)$. This is to say, for each $\alpha \; \varepsilon \; r(\lambda)$, $\alpha = \Sigma u_i a_{i\alpha}$ where $a_{i\alpha}$ is the αth component for each $i = 1, \ldots, r$ which means $r(\lambda)$ is finitely generated. (the u_i's do it).

Proposition 3.8': A direct product as well as a direct sum of divisible modules is divisible.

Proof: Let $\{A\gamma\}_{\gamma\varepsilon\Gamma}$ be a collection of divisible modules and let $A = \oplus A\gamma$ and $A' = \Pi A\gamma$. Let $a\varepsilon A$, $a = +_{a\alpha}$ such that $\ell(\lambda)\ a = 0$. Then $\ell(\lambda)\ a\gamma = 0$ for each γ so $a\gamma\ \varepsilon\ \lambda A\gamma$ as $A\gamma$ is divisible so $a\ \varepsilon \oplus \lambda A\gamma = \lambda + A\gamma = \lambda A$, so A is divisible.

Now let $a\varepsilon A'$, $a = (a\gamma)_{\gamma\varepsilon\Gamma}$ such that $\ell(\lambda)a = 0$. Then $\ell(\lambda)a\gamma = 0$ for each γ so $a\gamma\ \varepsilon\ \lambda A\gamma$ as $A\gamma$ divisible for each γ. So $a = (\lambda b\gamma)_{\gamma\varepsilon\Gamma}$ for some $b\gamma\ \varepsilon\ A\gamma$. So $a = \lambda(b\gamma)_{\gamma\varepsilon\Gamma}\ \varepsilon\ \lambda A'$ so A' is divisible.

Proposition 3.9: If R is a left PP-ring, then every right R-module possesses the "largest" torsion-free factor module, and every left R-module possesses the "largest" divisible submodule.

Proof: Let M be a right R-module. Then $R\lambda$ is a principal left ideal and since R is a left PP-ring, $R\lambda$ is projective. But then $\ell(\lambda)$ is finitely generated so the direct product of right torsion-free modules is torsion-free. Let $\{M/T_\alpha\}$ be the collection of all torsion-free factor modules of M. Then $\Pi_\alpha\ M/T_\alpha$ is torsion-free. But considering $M/\cap T_\alpha$ we see that $f\colon\ M/\cap T_\alpha \to \Pi_\alpha\ M/T$ defined by $x + \cap T_\alpha \to (x + T_\alpha)$ is an injection since if $(x + T_\alpha)\alpha = 0$ then $x\ \varepsilon\ T_\alpha$ for all α so $x\ \varepsilon \cap T_\alpha$ so $x + \cap T_\alpha = 0$. So we have $M/\cap T_\alpha$ is a submodule of $\Pi M/T_\alpha$ and by the mirror statement of the first part of Proposition 3.7 and the fact that PP $\to$ PF we have $M/\cap T_\alpha$ is torsion-free. Now we claim that $M/\cap T_\alpha$ is the largest torsion-free factor module of M. Since if M/T' is any factor module of M, $T'\ \varepsilon\ \{T_\alpha\}$ so $\cap T_\alpha \subset T'$ so M/T' is smaller than $M/\bigcap T_\alpha$. Let M be a left R-module and $\{M_\alpha\}$ be the collection of all divisible submodules of M. Then by Proposition 3.8' the direct sum, $\oplus M_\alpha$, is divisible but ΣM_α is a factor module of $\oplus M_\alpha$ as $f: \oplus M_\alpha \to \Sigma M_\alpha$ defined by $(x_\alpha)_{\alpha \varepsilon a} \to \sum_{\alpha \varepsilon a} x_\alpha$ is a surjection. So by the second part of Proposition 3.7, ΣM_α is divisible. Clearly ΣM_α is the largest divisible submodule.

Definition: A left module A will be called torsion if Hom (A, C) = 0 for every torsion-free module C.

<u>Proposition 3.10</u>: 1) The direct sum, $\oplus$ A, is a torsion module if and only if every summand A_α is a torsion module

2) If A is a torsion module, then so is any homomorphic image of A.

3) Any extension of a torsion module by a torsion module yields again a torsion module.

Proof 1): If $\oplus A_\alpha$ is a torsion module then Hom $(\oplus A_\alpha, C) = 0$ for all C torsion-free. ΠHom $(A_\alpha, C) \approx$ Hom $(\oplus A_\alpha, C) = 0$ for each α and all C torsion-free so for each α, A_α is torsion. Conversely if $\{A_\alpha\}$ is a collection of torsion modules Hom $(A_\alpha, C) = 0$ for each α and all C torsion-free) then $0 = \prod_\alpha$Hom (A_α, C) = Hom $(\oplus A_\alpha, C)$ for all C torsion free so $\oplus A_\alpha$ is torsion.

Proof 2) Let A' be a homomorphic image of A by f. $A \xrightarrow{f} A'$ and look at Hom (A', C) for any C torsion-free. Let $h \varepsilon$ Hom (A', C) and $x \varepsilon A'$. Then $x = f(a)$ for some $a \varepsilon A$ so $h(x) = h(f(a)) = (f \circ h)(a) = 0$ as $f \circ h$ is a homomorphism from $A \to C$ and A' is torsion.

Proof 3): Suppose we have $0 \to A \to B \to C \to 0$ with A and C torsion and suppose D is any torsion-free module. Then from the properties of Hom we get Hom (C, D) $\to$ Hom (B, D) $\to$ Hom (A, D) exact. But both ends are 0 as C and A torsion so Hom (B, D) = 0 and since D arbitrary, B is a torsion module.

Corollary 3.10: A module A has the largest torsion submodule.

Proof: Let $\{A_\alpha\}$ be the collection of all torsion submodules of A. Then $\oplus A_\alpha$ is torsion by the first part of the preceeding proposition. But ΣA_α is the homomorphic image of $\oplus A_\alpha$ as in the second part of Proposition 3.9 so by second part of Proposition 3.10, ΣA_α is torsion. Clearly it is the largest torsion submodule of A.

We call the largest torsion submodule of A the torsion submodule of A, and denote it by T(A).

A reduced module C is defined by the property that Hom (A, C) = 0 for every divisible module A.

Proposition 3.10': 1) The direct product ΠC_α is a reduced module if and only if every C_α is reduced.

2) If C is reduced, then so is any submodule of C.

3) Any extension of a reduced module by a reduced module yeilds again a reduced module.

Proof 1): If C_α is reduced then Hom $(A, \Pi C_\alpha) = 0$ for every divisible A. But Hom $(A, \Pi C_\alpha) = \Pi$Hom $(A, C_\alpha) = 0$ so Hom $(A, C_\alpha) = 0$ for each α and every divisible A. So C_α is reduced for each α. If for each α, C is reduced then Hom (A, C_α) = 0 for each divisible A. Then Hom (A, ΠC_α) = ΠHom (A, C_α)) = 0 for each divisible A so ΠC_α is reduced.

Proof 2): Let C' be a submodule of C. We get $0 \to C' \to C \to C/C' \to 0$ exact. Then for an A divisible we have $0 \to$ Hom (A, C') $\to$ Hom (A, C) exact. But Hom (A, C) = 0 so Hom (A, C') = 0 so C' is reduced.

Proof 3): Let $0 \to C \to C' \to C'' \to 0$ be exact with C and C' reduced. Let A be any divisible module. Then we have Hom (A, C) $\to$ Hom (A, C') $\to$ Hom (A, C") exact. Now both ends are 0 as C and C" are reduced so Hom (A, C') = 0 so C' is reduced.

<u>Corollary 3.10'</u>: Among submodules B of A with reduced factor modules there exists the smallest one, which we denote by D(A).

Proof: Let $\{B_\alpha\}$ be a collection of submodules of A such that A/B_α is reduced. Let B_α = D. Then A/D is reduced as $0 \to A/D \to \Pi A/B_\alpha$ is an injection, x + D $(x + B_\alpha)_\alpha$, so $A/D \subset \Pi A/B_\alpha$ which is reduced.

<u>Proposition 3.11</u>: If R is a PF ring, A is a torsion module if and only if it has only the trivial torsion-free factor A/A.

Proof: Let R be a PF ring and suppose A is a torsion module. Then a submodule of a torsion-free module is torsion free and Hom (A, C) = 0 for every torsion-free module C.

Suppose A has a non-trivial factor module A/B torsion-free. That is $B \neq A$. Then Hom (A, A/B) $\neq 0$ with A/B torsion-free so A not torsion which is a contradiction. Let R be a PF ring and suppose A is a module with only torsion-free factor A/A. Let T be a torsion-free module and let $\phi \in$ Hom (A, T). $\phi(A) \approx A/\text{Ker}\ \phi$ and $\phi(A)$ is torsion-free as it is a submodule of T. But this implies $A/\text{Ker}\ \phi = A/A$ that is $\text{Ker}\ \phi = A$ which means $\phi = 0$ so Hom (A, T) = 0 so A is torsion.

<u>Proposition 3.12</u>: If R is a PP ring, C is reduced if and only if it has only the trivial divisible submodule 0.

Proof: Let R be a PP ring and let C be reduced. Suppose C has a non-trivial divisible submodule B. Then Hom (B, C) $\neq 0$ as the inclusion map is in Hom (B, C). But this contradicts C being reduced, that is B = 0. Let R be a PP ring and let C be a module whose only divisible submodule is 0. Let B be a divisible module and let $\phi \in$ Hom (B, C). Then $\phi(B)$ is a submodule of C, but (B) $\approx B/\text{Ker}\ \phi$ a factor module of B hence divisible so $\phi(B)$ is a divisible submodule of C, hence $\phi(B) = 0$ so $\phi = 0$ so Hom (B, C) = 0. Since B was arbitrary, C is reduced.

If R is a commutative integral domain, our definitions of torsion modules and reduced modules coincide with the usual ones.

<u>Proposition 3.13</u>: Let R be a commutative integral domain. If Hom (A, C) = 0 for every torsion-free module C then for each $a \in A$ there exists $\lambda \in R$ such that $\lambda a = 0$.

Proof: Suppose there exists $a \varepsilon A$ such that $\lambda a \neq 0$ for any $\lambda \varepsilon R$, $\lambda \neq 0$. Then we get a homomorphism f: $R_a \to A$ and f injective. Also we have a map $Ra \to R \to I$ where I is the field of fractions of R so we have

$$\begin{array}{ccc} 0 \to Ra & \xrightarrow{f} & A \quad \text{since I injective} \\ i \downarrow & \swarrow \phi & \\ I & & \end{array}$$

So we have a map ϕ such that $\phi \circ f = i \neq 0$ so $\phi \neq 0$ and Hom (A, I) $\neq 0$ since I torsion-free A is not torsion-free.

<u>Proposition 3.14</u>: If for each $a \varepsilon A$ there exists $\lambda \varepsilon R$, $\lambda \neq 0$ such that $\lambda a = 0$ then Hom (A, C) = 0 for any torsion free C.

Proof: Let C be torsion-free and let f ε Hom (A, C) and $a \varepsilon A$, $A \neq 0$. Then there exists $\lambda \neq 0$, $\lambda \varepsilon R$ such that $\lambda a = 0$. f(a) = f(a) = 0 and since C torsion-free f(a) = 0 implies f(a) = 0 so f = 0 so Hom (A, C) = 0.

Let t(A) denote the set of torsion (Levy Sense) (5) elements of A.

<u>Lemma 3.15</u>: If R has the left quotient ring Q_1, t(A) is a submodule of A and coincides with the kernel of the natural mapping $A \to Q_1 \otimes A$.

Proof: t(A) is a submodule of A by Theorem 2.6. If $1 \otimes a = 0$ then $da = 0$ for some d regular in R as in the proof of Proposition 2.7. If a is Levy Sense torsion then $da = 0$ for some d regular in R so $1 \otimes a = d^{-1} \otimes da = d^{-1} \otimes 0 = 0$ in $Q_1 \otimes A$.

<u>Proposition 3.16</u>: For a (not necessarily commutative)

integral domain R, the following statements are equivalent.

1) R has the left quotient ring Q_ℓ

2) For any left module A, we have T(A) = t(A)

3) For any $\lambda \neq 0$, every element of $R/R\lambda$ is a torsion element.

Proof: 1 → 2: By Lemma 3.15 t(A) is a submodule of A and since it is torsion in the usual sense, hence also in Hattori's sense (3) and since T(A) is the largest Hattori sense torsion submodule of A, $t(A) \subset T(A)$. On the other hand, A/t(A) is torsion-free in the usual sense hence also in Hattori's sense since R has no zero divisors. So by definition Hom T(A), a/t(A) = 0, that is, $T(A) \subset t(A)$ so T(A) = t(A).

2 → 3: It is clear that Hom $(R/R\lambda, C) = 0$ for $\lambda \varepsilon R^*$ where R^* is the set of regular elements of R and for any torsion-free C, (Hattori's sense) that is, $R/R\lambda$ is a torsion module in Hattori's sense. If we let $A = R/R\lambda$ then T(A) = $R/R\lambda$, but T(A) = t(A) by 2 so $R/R\lambda = t(R/R\lambda)$, so each element of $R/R\lambda$ is a torsion element in Levy sense.

3 →1: Let $\lambda \varepsilon R^*$, that is, $\lambda \neq 0$, and let $u \varepsilon R$, then $U + R\lambda \varepsilon R/R\lambda$ hence torsion by 3 so there exists $\alpha \varepsilon R^*$ such that $\alpha u + R\lambda = 0$. That is $u \varepsilon R\lambda$ which means there exists $\beta \varepsilon R$, such that $\alpha u = \beta\lambda$. So for each $\lambda \varepsilon R^*$, $u \varepsilon R$ there exists $\alpha \varepsilon R^*$, $\beta \varepsilon R$, such that $\alpha u = \beta\lambda$ so R has a left quotient ring by Theorem 2.2.

If A is torsion-free then T(A) = 0 as Hom (T(A), A) = 0 implies T(A) = 0 since the identity map is a homomorphism from T(A) into A and the only way for the identity map to be

0 is for T(A) to be 0.

If T(A) = 0 A is not necessarily torsion-free as the following counter example shows.

Let $R = Z/4Z$ and let $A = 2Z/4Z$. Then $T(A) = 0$ but A is not torsion free as $\overline{2} \notin R$, $\overline{2}.\overline{2} = 0$ but $\overline{2} \varepsilon r(\overline{2})$ A as $r(\overline{2})A = \{\overline{0}\}$.

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